

A JORDAN DECOMPOSITION FOR GROUPS OF FINITE MORLEY RANK

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ABSTRACT. We prove a Jordan decomposition theorem for minimal connected simple groups of finite Morley rank with non-trivial Weyl group. From this, we deduce a precise structural description of Borel subgroups of this family of simple groups. Along the way we prove a Tetrachotomy theorem that classifies minimal connected simple groups. Some of the techniques that we develop help us obtain a simpler proof of a theorem of Burdges, Cherlin and Jaligot.

1. INTRODUCTION

This work is about *minimal connected simple groups* of finite Morley rank, in other words infinite simple groups of finite Morley rank whose proper definable connected subgroups are solvable. It is aspiring to offer a uniform treatment of this class of groups through a four-way categorization that uses a suitable notion of *Weyl group* which ultimately proves to be equivalent to all the proposed notions of Weyl groups for this class. Our approach, which follows a line closer to the theory of simple algebraic groups over algebraically closed fields, permits the introduction of notions of *semisimple* and *unipotent* elements using well-known concepts from group theory such as *Carter subgroups*, culminates in an abstract form of *Jordan decomposition* that carries all characterizing properties of the geometric one. This abstract approach turns out to be robust with respect to structural changes involving reducts. The techniques that we develop around the Weyl group analysis yield a new and simpler proof of a theorem on minimal connected simple groups of finite Morley rank.

The research on groups of finite Morley rank has progressed mostly around one main line, namely the analysis of the simple ones that are conjecturally isomorphic to linear algebraic groups over algebraically closed fields. This conjecture, in fact a natural question in the context of the model theory of algebraic structures, has served as a reference point in that any work on simple groups of finite Morley rank is an attempt to measure how far one is from a family of algebraic groups. These attempts have had recourse to two main sources in addition to model-theoretic foundations: the structure of linear algebraic groups, and finite group theory, especially the classification of the finite simple groups.

Historically, in the early stages of the analysis of infinite simple groups of finite Morley rank, most research dwelled on developing abstract analogues of concepts from algebraic group theory and proving theorems about these analogous to the known ones in the algebraic category. Nevertheless, the abstract context of groups of finite Morley rank falls short of providing an equivalent of the fine geometric information of algebraic groups, best observed through the use of such notions

as *unipotent* and *semisimple* elements. This deficiency as well as the increasing quantity of results closer in spirit to finite group theory, such as a satisfactory Sylow 2-theory, shifted most concentration towards the classification of the finite simple groups.

There are indeed deep parallels between the classification of the infinite simple groups of finite Morley rank and the classification of the finite simple groups. In particular, all active approaches are fundamentally inductive, the base case of the induction being the minimal connected simple groups. Consequently, minimal connected simple groups arise naturally if one considers a simple group of finite Morley rank whose proper definable connected simple sections are algebraic. Furthermore, arguments concerning minimal connected simple groups remain inspirational for more general partial classification results.

Unavoidably, methods and ideas from finite group theory have their limits as well. General Sylow theory is incomplete, elements of infinite order are frequently abundant and do not yield themselves easily to structural analysis. Moreover, and at least as fundamentally, ideas from finite group theory are too centered around 2-elements to allow a uniform treatment of various classes of infinite simple groups of finite Morley rank. This is far from satisfactory in the context of infinite simple groups of finite Morley rank where the most pathological examples, with very homogeneous structure, such as *bad groups*, or more generally groups of type (1) and (2) in Section 4 of the present article, do not have involutions, elements of order 2. Admittedly, this paper does not claim to shed more light on these groups that are not analyzable with known techniques. Nevertheless, there are various hypothetical simple non-algebraic groups of finite Morley rank with sufficiently versatile structure in that they have non-trivial Weyl groups, and hence have elements of finite order, such as those of type (3) in Section 4, that may not contain involutions. In such situations, the Jordan decomposition introduced in Section 8, equivalently the semisimple-unipotent dichotomy, offers a precise structural description as shown by the development that starts with the analysis of the centralizers of semisimple elements at the beginning of Section 8, and culminates in Theorems 9.12 and 9.15. The following excerpt from Theorem 9.12 provides a preview of the entire development:

Theorem 9.12. – *Let G be a group of finite Morley rank with non-trivial Weyl group. In each definable connected solvable subgroup H of G , the set H_u of unipotent elements is a definable connected subgroup such that $H = H_u \rtimes T$ for any maximal torus T of H .*

A noteworthy aspect of our attempt to describe the structure of definable connected solvable subgroups of minimal connected simple groups of finite Morley rank is that we have not been content with the known theory of solvable groups of finite Morley rank although evidently we have used it fully. Using the Jordan decomposition, we have systematically analyzed solvable groups definably embeddable in minimal connected simple groups. We have not been able to push far enough this newer approach so that we can eliminate any known difficult configurations such as the ones analyzed in [CJ04] and [Del08], but our methods place these works in a natural, uniform and general setting.

The attempt to characterize such geometric notions as unipotent and semisimple elements using only group-theoretic properties, naturally brought us to considering

the effects of structural changes on our techniques. Section 10 shows that our notions are robust with respect to reducts.

As the general setting of Sections 8 and 9 shows, a systematic analysis of Weyl groups is indispensable for our purposes. In Section 3, we make a thorough analysis of Weyl groups in minimal connected simple groups of finite Morley rank, an activity that had already reached a clear maturity in [BD09]. This ultimately yields the following theorem:

Theorem 3.13. – *Any non-nilpotent generous Borel subgroup B of a minimal connected simple group G is self-normalizing.*

The efforts invested in the analysis of Weyl groups have been also fruitful for the classification of the infinite simple groups of finite Morley rank. In section 11, we produce a simpler proof of one of the two main steps of the main result of [BCJ07], which is the following theorem:

Theorem 11.1. – *If G has odd type and Prüfer 2-rank at least two, then G has no strongly embedded subgroup.*

We expect that the more conceptual methods employed here will generalize more easily to approaches towards new classification results.

2. A CRASH COURSE ON GROUPS OF FINITE MORLEY RANK

This section exists mainly for the convenience of our readers who, given the nature of the pursued approaches, may include specialists not familiar with groups of finite Morley rank. We will start from the most fundamental and elementary aspects of model theory, and develop a quick introduction to the theory of groups of finite Morley rank that is relevant to this article. As a result, this section can be used as an introduction to groups of finite Morley rank or as reference. In particular, any reader familiar with these subjects can skip it.

Morley rank is one of the many dimension notions in model theory. It generalizes the notion of *Zariski dimension* of closed sets in algebraic geometry over algebraically closed fields, and as every notion of dimension introduced in any geometric theory, it allows to develop a theory of independence.

In algebraic geometry, closed sets are assigned a dimension. In the case of a structure that admits Morley rank, *definable* sets are those that yield themselves to the measurement by the Morley rank. This measurement is done by keeping the combinatorial content of the Zariski dimension. Before detailing how this is done, we will go over some fundamental concepts of model theory.

A *structure* \mathcal{M} is an underlying set M , called sometimes the *universe* of \mathcal{M} , equipped with

- a possibly empty family $\{c_i | i \in I_C\}$ of distinguished elements of M , called *constants*;
- a possibly empty family $\{f_i | i \in I_F\}$ of *functions* with $f_i : M^{n_i} \rightarrow M$ for each $i \in I_F$, where $n_i \in \mathbb{N}^*$ and depends only on f_i ;
- a family $\{R_i | i \in I_R\}$ of relations on M^{k_i} for each $i \in I_R$, where $k_i \in \mathbb{N}^*$ and depends only on R_i .

The three mutually disjoint families of indices I_C , I_F , I_R , and the correspondance that associates an index to a constant, function or relation respectively is called the *signature* of \mathcal{M} . The n_i and the k_i are the *arities* of the functions and relations

respectively. It is worth noting that the equality is always part of the relations, the reason why the family of relations is never empty. Also, constants are nothing but 0-ary functions.

To concretize this formalism, a group can be regarded as the following structure

$$\mathcal{G} = (G; ., {}^{-1}, 1, =)$$

where G is the underlying non-empty set, $.$ is the binary group operation, the unary function ${}^{-1}$ is the group inversion, 1 is the identity element of the group \mathcal{G} and $=$ is the only relation. It is common practice to exclude the equality from the notation.

Various facts about a structure can clearly be expressed, subsets of cartesian powers of the underlying set be defined using the members of the signature. For example, “ $x.y = y.x$ ” expresses that x and y commute in a group, an expression that can be strengthened with some “quantification” to express the center of a group.

A *first-order language* is the formalism consisting of symbols that name members of the signature of a fixed structure, variable symbols, quantifiers, logical connectives, and a set of inductively defined syntactic rules to juxtapose these symbols. It thus describes what sets can be defined using a fixed structure. We will not go over the details of the formalism of first-order structures but emphasize two necessary conditions for first-order languages: any acceptable string of symbols, a *well-formed formula*, is of finite length; only variables are quantified.

One can for example fix a group $\mathcal{G} = (G; ., {}^{-1}, 1)$ seen through the “language of groups”, $\mathcal{L} = \{., {}^{-1}, 1\}$, and define its center as the elements satisfying the first-order formula $\forall y \, xy = yx$. Or one can see the same group as an “enriched” or *expanded* structure $\mathcal{G}^+ = (G; ., {}^{-1}, 1, g)$, where g is a constant symbol naming a particular element and define within the first-order context the centralizer of g as the set of elements of G satisfying the well-formed formula $x.g = g.x$ written in the language $\{., {}^{-1}, 1, g\}$. On the other hand, even if there were enough many symbols in the language, it may not be possible to express the centralizer of an infinite set using a well-formed formula. One will need alternative definitions to conjoining infinitely many formulas of the form $x.g = g.x$. An extreme but useful example of such an alternative is encountered in an abelian groups: the formula $x = x$ suffices to express the centralizer of an element.

The preceding sequence of examples brings us to the fundamental notion of a *definable set*. A subset of a Cartesian power of the underlying set of a fixed structure \mathcal{M} is said to be *definable in \mathcal{M}* if its elements can be described using a first-order formula. Here, it should be emphasized that there is no ambiguity as to the choice language since this is completely determined by the signature of \mathcal{M} . In the same vein, a function or relation is definable if its graph is a definable set. Using these notions, one extends the notion of definability, and introduces a structure that is definable in another structure. Intuitively speaking, a structure \mathcal{M} is said to be definable in a structure \mathcal{M}' if its underlying set and signature are definable in \mathcal{M}' . This definition is extended further by allowing “quotients”, in other words definable sets modulo *definable equivalence relations*. Some call these structures *interpretable*. We will keep using the word “definable” since in a suitable model-theoretic setup everything interpretable becomes definable.

To concretize the preceding definitions, a good example in our context is the notion of a definable quotient space in a group. Indeed, in a group structure \mathcal{G} with underlying set G , a definable subgroup H induces a definable quotient, namely

G/H since the equivalence relation of belonging to the same coset is definable as soon as H is; G/H is interpretable in G .

A second relevant group-theoretic example is an algebraic group over a field. By its very definition, the underlying set of such a group, its group operations and identity element are all definable using field operations. On the other hand, whether one can recover up to a reasonable isomorphism, the underlying field and its geometry using the bare group structure is a less obvious question. Indeed, the answer may even be negative, and the quest for such an answer is a major activity in model theory that lies among the sources of motivation for this paper as well.

We give one final example of definability of relevance for this paper. It is related to the notion of *expansion* of a structure. Indeed, one can start with a structure \mathcal{M} , then increase its signature without changing the underlying set M . The expanded structure \mathcal{M}^+ is an expansion of \mathcal{M} , and \mathcal{M} , a *reduct* of \mathcal{M}^+ , is definable in \mathcal{M}^+ . One can expand a structure by enriching any part of the signature, not just constants. In section 10, we will further analyze the impacts of reducts of groups on various notions introduced in this paper such as semisimple and unipotent elements.

With the notion of definable set at hand, we can introduce the Morley rank of a definable set. We start by fixing a structure \mathcal{M} . A definable set A in \mathcal{M} , which may be a subset of any cartesian power of the underlying universe M , is of rank *at least* $\alpha + 1$, where α is an ordinal, if there exists an infinite family $\{A_i \mid i \in I\}$ of mutually disjoint definable subsets of A each of which is of rank at least α . For limit ordinals, one takes the limit. The set A is said to be of rank α if it is at least of rank α and it is not of rank greater than or equal to $\alpha + 1$. The Morley rank of a structure is the rank of the set defined by $x = x$. We should mention that this 1-dimensional definition implies that all cartesian powers of the given structure, defined by $\bigwedge_{i=1}^k x_i = x_i$ ($k \in \mathbb{N}$) admit Morley rank though the relationships among actual numerical values of the ranks may be different from the expected ones. As this definition shows, Morley rank is an ordinal valued dimension. Nevertheless, we will analyze only structures of *finite* Morley rank. This definition implies that all finite structures are of Morley rank 0. We will mostly be interested in infinite structures.

We will note the Morley rank of a definable set X by $\text{RM}(X)$. To be more precise, it is the Morley rank of a formula in a fixed language. This may correspond to different sets when one goes to *elementary extensions*. Indeed the definition in the preceding paragraph is insufficient in general. In order to obtain a robust notion of dimension, one has to consider a structure together with its ω -saturated *elementary extensions*. Nevertheless, it is a theorem of Poizat in [PoizGrSt] that this is not necessary in the case of a group of finite Morley rank. Thus, we will not speak about elementary extensions nor saturation.

A group of finite Morley rank has additional nice properties. We mention two of them:

- (i) If $f : A \longrightarrow B$ is a definable function between two sets definable in a group of finite Morley rank then the set

$$\{ b \in B \mid \text{RM}(f^{-1})(b) = i \}$$

is definable.

- (ii) If $f : A \rightarrow B$ is a definable function between two sets definable in a group of finite Morley rank such that the fibers are all of the same rank n , then $\text{RM}(A) = \text{RM}(B) + n$.

The above properties of groups of finite Morley rank are clearly reminiscent of the behaviour of the Zariski dimension in algebraic groups over algebraically closed fields. Indeed, if K is an algebraically closed field of a certain characteristic, then it can be shown that the subsets of K definable in the field structure $\mathcal{K} = (K; +, \cdot, ^{-1}, -, 0, 1)$ are exactly the finite and cofinite ones. This shows that \mathcal{K} is of Morley rank 1. Moreover, as already intuitively expected, a structure definable in a structure of finite Morley rank is of finite Morley rank. Thus algebraic groups over algebraically closed fields are examples of groups of finite Morley rank. In fact, to this day they form the largest class of known algebraically interesting examples of groups of finite Morley rank. The following central conjecture in the analysis of groups of finite Morley rank, which ties in with many different general model-theoretic questions, can also be regarded as an attempt to explain the ubiquity of algebraic groups:

Algebraicity Conjecture (Cherlin-Zil’ber): *An infinite simple group of finite Morley rank, seen as a pure group structure, is a linear algebraic group over an algebraically closed field.*

In stating this conjecture, we have taken pains to emphasize the “purity” of the group, in that as a structure the conjecture is about “pure groups”, in other words group structures of the form $\mathcal{G} = (G; \cdot, ^{-1}, 1)$. Nevertheless, it is common practice in model theory to call “a group of finite Morley rank” any group definable in a structure of finite Morley rank; or more generally, to mean by a “group” a group that is a reduct of a richer structure. As we will shortly see this does not cause any ambiguity for the Algebraicity Conjecture as simplicity is not affected by changes in definability.

Before going any further, we find it appropriate to justify the appearance of algebraically closed fields. The following, together with Fact 2.6 below, is one of the two oldest results on algebraic structures of finite Morley rank:

Fact 2.1. – [Mac71Fi, Theorem 1] [BN94, Theorem 8.1] *A field definable in a structure of finite Morley rank is either finite or algebraically closed.*

The ordinal character of the Morley rank forces a group of finite Morley rank to satisfy strong finiteness conditions, the most fundamental being the *descending chain condition on definable subgroups*: in a group of finite Morley rank, there is no infinite descending chain of definable subgroups. This property allows one to introduce various notions in the abstract context of groups of finite Morley rank, analogous to geometric aspects of algebraic groups. Thus, the *connected component* of a group G of finite Morley rank, noted G° and defined as the smallest definable subgroup of finite index, does exist and is the intersection of all definable subgroups of finite index in G . A group of finite Morley rank is said to be *connected* if it is equal to its connected component.

The connected component of a group is an example of a “large” definable set in that it is of the same rank as the ambient group. In general, a definable subset X of G is said to be *generic* if $\text{RM}(X) = \text{RM}(G)$. Intuitively speaking, a connected group is one where generic subsets intersect generically.

In a dual vein, if X is an arbitrary subset of a group G of finite Morley rank, then one defines its *definable hull*, noted $d(X)$ as the intersection of all definable subgroups of G containing X . Thanks to the descending chain condition, the definable hull of a set is well-defined and offers an analogue of the Zariski closure in algebraic geometry. The existence of a definable hull allows to speak about the connected component of an arbitrary subgroup of the ambient group G : if X is subgroup, then X° is defined as $X \cap d(X)^\circ$, and X is said to be connected if $X = X^\circ$. It is worth noting that the notion of definable hull has proven to be very effective in illuminating the algebraic structure of groups of finite Morley rank since many algebraically interesting subgroups such as Sylow subgroups, are not definable. Moreover, various algebraic properties are preserved as one passes to the definable hull:

Fact 2.2. – (Zil’ber) [BN94, Corollary 5.38] *Let G be a group of finite Morley rank and H be a solvable (resp. nilpotent) subgroup of class n . Then $d(H)$ has the same properties.*

Another fundamental notion that also has connections with definability and connectedness is that of an *indecomposable set*. A definable set in a group G of finite Morley rank is said to be indecomposable if for any definable subgroup $H \leq G$ whenever cosets of H decompose X into more than one subset, then they decompose into infinitely many. In particular, an indecomposable subgroup is a connected subgroup.

The notion of indecomposable set, that has analogues well-known to algebraic group theorists, is of fundamental importance in that it helps clarify the definable structure of a group of finite Morley rank. This mostly due to the *Zil’ber’s indecomposability theorem* which states that indecomposable sets which contain the identity element of the group generate definable connected subgroups. We will use its following corollaries frequently, mostly without mention:

Fact 2.3. – [BN94, Corollary 5.28] *Let G be a group of finite Morley rank. Then the subgroup generated by a family of definable connected subgroups of G is definable and the setwise product of finitely many of them.*

Fact 2.4. – [BN94, Corollaries 5.29 and 5.32] *Let G be a group of finite Morley rank.*

- (1) *Let $H \leq G$ be a definable connected subgroup of G and X an arbitrary subset of G . Then the subgroup $[H, X]$ is definable and connected.*
- (2) *Let H be a definable subgroup of G . Then the members of the derived $(H^{(n)})$ and lower central series (H^n) of H are definable. If H is connected, then so are these subgroups of H .*

Zil’ber’s indecomposability theorem has another consequence that is of relevance in the context of this article and to which we have already alluded: a group of finite Morley rank is simple if and only if it has no definable, normal, proper, non-trivial subgroup. This remarkable consequence is relevant for section 10.

The algebraic structure of an arbitrary group of finite Morley rank naturally exhibits similarities to that of a linear algebraic group. A group of finite Morley rank is built up from definable, minimal subgroups that are abelian:

Fact 2.5. – [Rei] [BN94, Theorem 6.4] *In a group of finite Morley rank, a minimal, infinite, definable subgroup A is abelian. Furthermore, either A is divisible or is an elementary abelian p -group for some prime p .*

This simple and historically old fact is what permits many inductive arguments using Morley rank. The additional structural conclusions in Fact 2.5 are related to the following general structural description of abelian groups of finite Morley rank.

Fact 2.6. – [Mac70Gr, Theorems 1 and 2] [BN94, Theorem 6.7] *Let G be an abelian group of finite Morley rank. Then the following hold:*

- (1) $G = D \oplus C$ where D is a divisible subgroup and C is a subgroup of bounded exponent;
- (2) $D \cong \bigoplus_p \text{prime}(\bigoplus_{I_p} \mathbb{Z}_{p^\infty}) \oplus \bigoplus_I \mathbb{Q}$ where the index sets I_p are finite;
- (3) $G = DB$ where D and B are definable characteristic subgroups, D is divisible, B has bounded exponent and $D \cap B$ is finite. The subgroup D is connected. If G is connected, then B can be taken to be connected.

It easily follows from this detailed description of abelian groups of finite Morley rank that, in general, groups of finite Morley rank enjoy the property of *lifting torsion from definable quotients*. More precisely, if G is a group of finite Morley rank, $H \leq G$ a definable subgroup of G and $g \in G$ such that $g^n \in H$ for some $n \in \mathbb{N}^*$, where n is assumed to be the order of g in $d(g)/d(g) \cap H$ and is a π -number with π a set of prime numbers, then there exists $g' \in gH \cap d(g)$ such that g' is again a π -element. Here, a π -number is a natural number whose prime divisors belong to π , and a π -element is an element whose order is a π -number. One important point where this elementary but important property will be of crucial use is the analysis of Weyl groups in Section 3. The torsion-lifting property will be used without mention.

Fact 2.6 was later generalized to the context of nilpotent groups of finite Morley rank using techniques of algebraic character:

Fact 2.7. – [Nes91, Theorem 2] [BN94, Theorem 6.8 and Corollary 6.12] *Let G be a nilpotent group of finite Morley rank. Then G is the central product $B * D$ where D and B are definable characteristic subgroups of G , D is divisible, B has bounded exponent. The torsion elements of D are central in G .*

The structural description provided by Facts 2.6 and 2.7 can be regarded as a weak “Jordan decomposition” in groups of finite Morley rank since, using the notation of the fact, B and D are respectively abstract analogues of unipotent and semisimple parts of a nilpotent algebraic group. This viewpoint is indeed weak in that when $B = 1$ and D is a torsion-free group, it is not possible to decide whether D is semisimple or unipotent (characteristic 0).

The description of the divisible nilpotent groups of finite Morley rank can be refined further:

Fact 2.8. – [Nes91, Theorem 3] [BN94, Theorem 6.9] *Let G be a divisible nilpotent group of finite Morley rank. Let T be the torsion part of G . Then T is central in G and $G = T \oplus N$ for some torsion-free divisible nilpotent subgroup N .*

This description has been extensively exploited in most works on groups of finite Morley rank and this paper is no exception to this. Remarkably, as will be explained later in this section, and used later in this paper, a finer analysis of nilpotent groups

of finite Morley rank, even when torsion elements are absent, is possible using a suitable notion of unipotence.

We also include the following two elementary properties of nilpotent groups of finite Morley rank that generalize similar well-known properties of algebraic groups. Other similarities involving normalizer conditions will be mentioned later in this section in the context of the finer unipotent analysis.

Fact 2.9. –

- (1) [BN94, Lemma 6.3] *Let G be a nilpotent group of finite Morley rank and H a definable subgroup of infinite index in G . Then $N_G(H)/H$ is infinite.*
- (2) [BN94, Exercice 6.1.5] *Let G be a nilpotent group of finite Morley rank. Any infinite normal subgroup has infinite intersection with $Z(G)$.*

As in many other classes of groups, there is a long way between nilpotent and solvable groups of finite Morley rank. Remarkably, the differences are best measured by field structures that are definable in solvable non-nilpotent groups of finite Morley rank. All the results that we will need in this paper about solvable groups illustrate this “definably linear” aspect of solvable groups of finite Morley rank. The most fundamental one is the following:

Fact 2.10. – (Zil’ber) [BN94, Theorem 9.1] *Let G be a connected, solvable, non-nilpotent group of finite Morley rank. Then there exist a field K and definable connected sections U and T of G' and G/G' respectively such that $U \cong (K, +)$, and T embeds in (K^\times, \cdot) . Moreover, these mappings are definable in the pure group G , and each element of K is the sum of a bounded number of elements of T . In particular, K is definable in G and hence of finite Morley rank.*

The ability to define an algebraically closed field in a connected solvable eventually culminates in the following result that generalizes a well-known property of connected solvable algebraic groups.

Fact 2.11. – [BN94, Corollary 9.9] *Let G be a connected solvable group of finite Morley rank. Then G' is nilpotent.*

This result is related to more group theoretic notions using properties of groups of finite Morley rank. The *Fitting subgroup* of a group of finite Morley rank G , noted here $F(G)$, is defined to be the maximal, definable, normal, nilpotent subgroup of G . Thanks to the works of Belegarde and Nesin, this definition turns out to be equivalent to the one used in finite group theory: the subgroup generated by all normal, nilpotent subgroups. The following result of Nesin shows that the Fitting subgroup shares properties of its unipotent analogues in algebraic groups. This is yet another consequence of the linear behaviour of solvable groups of finite Morley rank of which various refinements have been obtained first in the works of Altseimer and Berkman, later of the third author.

Fact 2.12. – [BN94, Theorem 9.21] *Let G be a connected solvable group of finite Morley rank. Then $G/F(G)^\circ$, thus $G/F(G)$ are divisible abelian groups.*

Beyond solvable?... Since this paper is about minimal connected simple groups of finite Morley rank and we already mentioned examples that motivate the Algebraicity Conjecture, at this point we will be content with the most extreme minimal counterexample whose existence is a major open problem, namely *bad groups*. By definition a bad group is a connected, non-solvable, group of finite Morley rank

whose proper definable connected subgroups are nilpotent. One easily shows that if a bad group exists, then there exists a simple one. In particular, such a group is minimal, connected and simple. The following make up for most of the few but striking known properties of simple bad groups.

Fact 2.13. – [BN94, Theorem 13.3] *Let G be a simple bad group. Then the following hold:*

- (1) *The Borel subgroups of G are conjugate.*
- (2) *Distinct Borel subgroups of G intersect trivially.*
- (3) *G is covered by its Borel subgroups.*
- (4) *G has no involutions.*
- (5) *$N_G(B) = B$ for any Borel subgroup B of G .*

A Borel subgroup of a group of finite Morley rank is a maximal, definable, connected, solvable subgroup.

As mentioned above, very little is known about bad groups. Clearly, the stated properties are far from those of simple algebraic groups. Except for the primes 2 and 3, it is not even known whether a simple bad group can be of prime exponent. This is the main reason why below we will be careful while treating p -subgroups of groups of finite Morley rank.

In this paper, for each prime p , a Sylow p -subgroup of any group G is defined to be a maximal *locally finite* p -subgroup. By Fact 2.14 (1), such a subgroup of a group of finite Morley rank is nilpotent-by-finite.

Fact 2.14. –

- (1) [BN94, Theorem 6.19] *For any prime number p , a locally finite p -subgroup of a group of finite Morley rank is nilpotent-by-finite.*
- (2) [BN94, Proposition 6.18 and Corollary 6.20] *If P is a nilpotent-by-finite p -subgroup of a group of finite Morley rank, then $P^\circ = B * T$ is the central product of a definable, connected, subgroup B of bounded exponent and a divisible abelian p -group. In particular, P° is nilpotent.*

The assumption of local finiteness for p -subgroups is rather restrictive but unavoidable as was implied by the remarks after Fact 2.13. The only prime for which the mere assumption of being a p -group is equivalent to being a nilpotent-by-finite in groups of finite Morley rank is 2. The prime 2 is also the only one for which a general Sylow theorem is known for groups of finite Morley rank:

Fact 2.15. – [BN94, Theorem 10.11] *In a group of finite Morley rank the maximal 2-subgroups are conjugate.*

Before reviewing the Sylow theory in the context of solvable groups where it is better understood, we introduce some terminology related to the unipotent/semisimple decomposition, as well as some of its implications for the analysis of simple groups of finite Morley rank. For each prime p , a nilpotent definable connected p -group of finite Morley rank is said to be *p -unipotent* if it has bounded exponent while a *p -torus* is a divisible abelian p -group.

In general, a p -torus is not definable but enjoys a useful finiteness property in a group of finite Morley rank. It is the direct sum of finitely many copies of \mathbb{Z}_{p^∞} , the Sylow p -subgroup of the multiplicative group of complex numbers. In particular, the p -elements of order at most p form a finite elementary abelian p -group of which

the rank is called the *Prüfer p -rank* of the torus in question. Thus, in any group of finite Morley rank where maximal p -tori are conjugate, the Prüfer p -rank of the ambient group is defined as the Prüfer p -rank of a maximal p -torus.

The choice of terminology, “unipotent” and “torus”, is not coincidental. Fact 2.14 (2) shows that the Sylow p -subgroups of a group of finite Morley rank have similarities with those of algebraic groups. These are of bounded exponent when the characteristic of the underlying field is p , and divisible abelian when this characteristic is different from p . In the notation of Fact 2.14 (2), this case division corresponds to $T = 1$ or $B = 1$ respectively when the Sylow p -subgroup in question is non-trivial.

A similar case division for the prime 2 has played a major role in developing a strategy to attack parts of the Cherlin-Zil’ber conjecture. In this vein, a group of finite Morley rank is said to be of *even type* if its Sylow 2-subgroups are infinite of bounded exponent ($B \neq 1$, $T = 1$), of *odd type* if its Sylow 2-subgroups are infinite and their connected components are divisible ($B = 1$, $T \neq 1$), of *mixed type* if $B \neq 1$ and $T \neq 1$ and of *degenerate type* if they are finite.

The main result of [ABC08] states that a simple group of finite Morley rank that contains a non-trivial unipotent 2-subgroup is an algebraic group over an algebraically closed field of characteristic 2. In particular, there exists no simple group of finite Morley rank of mixed type. In this article, we will use this result and refer to it as the *classification of simple groups of even type*. Despite spectacular advances for groups of odd type, no such extensive conclusion has been achieved. In the degenerate type, it has been shown in [BBC07] that a connected group of finite Morley rank of degenerate type has no involutions:

Fact 2.16. – [BBC07, Theorems 1 and 3] *Let G be a connected group of finite Morley rank whose maximal p -subgroups are finite. Then G contains no elements of order p .*

The following generalization of a well-known semisimple torsion property of algebraic groups was proven following a similar line of ideas.

Fact 2.17. – [BC08b, Theorem 3] *Let G be a connected group of finite Morley rank, π a set of primes, and a any π -element of G such that $C_G(a)^\circ$ does not contain a non-trivial π -unipotent subgroup. Then a belongs to any maximal π -torus of $C_G(a)$.*

As was mentioned above, the Sylow theory is much better understood in solvable groups of finite Morley rank. This is one reason why one expects to improve the understanding of the structure of minimal connected simple groups of finite Morley rank although even in the minimal context additional tools are indispensable. We first review the parts of what can now be called the classical Hall theory for solvable groups of finite Morley rank that are relevant for this paper. Then we will go over more recent notions of tori, unipotence and Carter theory as was developed in the works Cherlin, Deloro, Jaligot, the second and third authors.

One now classical result on maximal π -subgroups of solvable groups of finite Morley rank is the Hall theorem for this class of groups:

Fact 2.18. – [BN94, Theorem 9.35] *In a solvable group of finite Morley rank, any two Hall π -subgroups are conjugate.*

Hall π -subgroups are by definition maximal π -subgroups. The Hall theorem was motivated by finite group theory while the next two facts have their roots in the structure of connected solvable algebraic groups:

Fact 2.19. –

- (1) [BN94, Corollary 6.14] *In a connected nilpotent group of finite Morley rank, the Hall π -subgroups are connected.*
- (2) [BN94, Theorem 9.29] [Fré00a, Corollaire 7.15] *In a connected solvable group of finite Morley rank, the Hall π -subgroups are connected.*

We also recall the following easy but useful consequence of Fact 2.12.

Fact 2.20. – *A solvable group of finite Morley rank G has a unique maximal p -unipotent subgroup.*

On the toral side, we will need the following analogue of well-known properties of solvable algebraic groups:

Fact 2.21. – [Fré00b, Lemma 4.20] *Let G be a connected solvable group of finite Morley rank, p a prime number and T a p -torus. Then $T \cap F(G) \leq Z(G)$.*

More recent research and ideas oriented towards the understanding the nature of a generic element of a group of finite Morley rank have given rise to two important notions of tori. A divisible abelian group G of finite Morley rank is said to be: a *decent torus* if $G = d(T)$ for T its (divisible) torsion subgroup; a *pseudo-torus* if no definable quotient of G is definably isomorphic to K_+ for an interpretable field K .

The following remark based on important work of Wagner on bad fields of non zero characteristic was the first evidence of the relevance of these notions of tori.

Fact 2.22. – [AC04, Lemma 3.11] *Let F be a field of finite Morley rank and nonzero characteristic. Then F^\times is a good torus.*

A good torus is a stronger version of a decent torus in that the defining property of a decent torus is assumed to be hereditary.

Using the geometry of groups of finite Morley rank provided by genericity arguments that we will outline later in this section, Cherlin and later the third author obtained the following conjugacy results. It is worth mentioning that such results were possible mainly because one can describe the generic element of a group of finite Morley rank. This is the case when a group of finite Morley rank has non-trivial decent or pseudo-tori, as well as it is the case when it has generous Carter subgroups as witnessed by Fact 2.40 (1) below.

Fact 2.23. –

- (1) [Che05, Extended nongenericity] *In a group of finite Morley rank, maximal decent tori are conjugate.*
- (2) [Fré09, Theorem 1.7] *In a group of finite Morley rank, maximal pseudo-tori are conjugate.*

Below, we include several facts about decent and pseudo-tori mostly for the practical reason that we will need them. Nevertheless, they have the virtue of justifying that these more general notions of tori, introduced to investigate more efficiently the structure of groups of finite Morley rank, share crucial properties of tori in algebraic groups, and thus illuminating what aspects of a notion of algebraic torus influence the structure of algebraic groups.

Fact 2.24. –

- (1) [Fré06b, Lemma 3.1] *Let G be a group of finite Morley rank, N be a normal definable subgroup of G , and T be a maximal decent torus of G . Then TN/N is a maximal decent torus of G/N and every maximal decent torus of G/N has this form.*
- (2) [Fré09, Corollary 2.9] *Let G be a connected group of finite Morley rank. Then the maximal pseudo-torus of $F(G)$ is central in G .*
- (3) [AB09, Theorem 1] *Let T be a decent torus of a connected group G of finite Morley rank. Then $C_G(T)$ is connected.*
- (4) [Fré09, Corollary 2.12] *Let T be a pseudo-torus of a connected group G of finite Morley rank. Then $C_G(T)$ is connected and generous in G , and $N_G(C_G(T))^\circ = C_G(T)$.*

So far, we have emphasized notions of tori and their generalizations in groups of finite Morley rank. Before moving to the unipotent side, it is necessary to go over a notion that is related to both sides and thus fundamental to the understanding of groups of finite Morley rank: *Carter subgroups*. In groups of finite Morley rank, Carter subgroups are defined as being the definable connected nilpotent subgroups of finite index in their normalizers. We summarize the main results concerning these subgroups in Fact 2.25.

In an algebraic group, Carter subgroups correspond to maximal tori. Hence, the notion of Carter subgroup offers a possibility to approach properties of algebraic tori in a purely group-theoretic form. Carter subgroups have strong ties with the geometry of groups of finite Morley rank stemming from genericity arguments. We will review some of these connections later in this section around Fact 2.40.

Fact 2.25. – *Let G be a group of finite Morley rank.*

- (1) [FJ05], [FJ08, Theorem 3.11] *G has a Carter subgroup.*
- (2) [Fré09, Corollary 2.10] *Each pseudo-torus is contained in a Carter subgroup of G .*
- (3) [Wag94, Theorem 29] *If G is solvable, its Carter subgroups are conjugate.*
- (4) [Fré08, Theorem 1.2] *If G is a minimal connected simple group, its Carter subgroups are conjugate.*
- (5) [Fré00a, Théorèmes 1.1 and 1.2] *If G is connected and solvable, any subgroup of containing a Carter subgroup of G is definable, connected and self-normalizing.*
- (6) [Fré00a, Corollaire 5.20], [FJ08, Corollary 3.13] *If G is connected and solvable, for each normal subgroup N , Carter subgroups of G/N are exactly of the form CN/N , with C a Carter subgroup of G .*
- (7) [Fré00a, Corollaire 7.7] *Let G be a connected solvable group of class 2 and C be a Carter subgroup of G . Then there exists $k \in \mathbb{N}$ such that $G = G^k \rtimes C$.*

The notion of *abnormality* is tightly connected to that of a Carter subgroup in solvable group theory. In the context of solvable groups of finite Morley rank, abnormal subgroups of solvable groups were analyzed in detail in [Fré00a]. By definition, a subgroup H of any group G is said to be *abnormal* if $g \in \langle H, H^g \rangle$ for every $g \in G$. In a connected solvable group of finite Morley rank abnormal subgroups turn out to be definable and connected. Their relation to Carter subgroups is as follows:

Fact 2.26. –

- (1) [Fré00a, Théorème 1.1] *In a connected solvable group of finite Morley rank, a definable subgroup is a Carter subgroup if and only if it is a minimal abnormal subgroup.*
- (2) [Fré00a, Théorème 1.2] *Let G be a connected solvable group of finite Morley rank, and H be a subgroup of G . Then the following are equivalent:*
 - (i) *H is abnormal;*
 - (ii) *H contains a Carter subgroup of G .*

An important class of abnormal subgroup is formed by *generalized centralizers*. If G is an arbitrary group, A a subgroup and $g \in N_G(A)$, then the generalized centralizer of g in A is defined by $\{x \in A \mid \text{il existe } n \in \mathbb{N} \text{ tel que } [x, {}_n g] = 1\}$. Let us remind that $[x, {}_0 g] = x$ and $[x, {}_{n+1} g] = [[x, {}_n g], g]$ for every $n \in \mathbb{N}$. More generally, if $Y \subseteq N_G(A)$ then $E_A(Y) = \cap_{y \in Y} E_A(y)$.

In general, a generalized centralizer need not even be a subgroup. On the other hand, in a connected solvable group of finite Morley rank, it turns out to be a definable, connected subgroup that sheds considerable light on the structure of the ambient group:

Fact 2.27. – [Fré00a, Corollaire 7.4] *Let G be a connected solvable group of finite Morley rank and H be a nilpotent subgroup of G . Then $E_G(H)$ is abnormal in G .*

In addition to the information they provide, the generalized centralizers are in a sense more practical tools than the centralizers of sets. This is mainly because a generalized centralizer contains the elements that they “centralize”, and this containment is rather special:

Fact 2.28. – [Fré00a, Corollaire 5.17] *Let G be a connected solvable group of finite Morley rank and H a subset of G that generates a locally nilpotent subgroup. Then $E_G(H) = E_G(d(H))$, is definable, connected, and H is contained in $F(E_G(H))$. In particular, $d(H)$ is nilpotent and the set of nilpotent subgroups of G is inductive.*

Thus generalized centralizers provide definable connected enveloping subgroups for arbitrary subsets of connected solvable groups of finite Morley rank.

The notion of a p -unipotent group gives a robust analogue of a unipotent element in an algebraic group over an algebraically closed field of characteristic p . As was mentioned after Fact 2.7 however, there is no such analogue for unipotent elements in characteristic 0, and this has been a major question to which answers of increasing levels of efficiency have been given. The first step in this direction can be traced back to the notion of *quasiunipotent radical* introduced in unpublished work by Altseimer and Berkman. This notion is still of relevance, and we will use a refinement of Fact 2.12 proven by the third author using the notion of the notion of quasiunipotent radical.

A definable, connected, nilpotent subgroup of group G of finite Morley rank is said to be *quasi-unipotent* if it does not contain any non-trivial p -unipotent subgroup. The quasi-unipotent radical of a group of finite Morley rank G , noted $Q(G)$, is the subgroup generated by its quasi-unipotent subgroups. By Fact 2.3, $Q(G)$ is a definable, connected subgroup. Clearly, $Q(G) \triangleleft G$. Less clearly, though naturally, the following is true:

Fact 2.29. – [Fré00b, Proposition 3.26] *Let G be connected solvable group of finite Morley rank. Then $G/Q(G)$ is abelian and divisible.*

The notions of *reduced rank* and $U_{0,r}$ -groups were introduced by the second author in order to carry out an analogue of local analysis in the theory of the finite simple groups. In a similar vein, a theory of *Sylow $U_{0,r}$ -subgroups* was developed. The notion of *homogeneity* was introduced by the third author in his refinement of the unipotence analysis. We summarize these in the following definition:

Definition 2.30. – [Bur04], [Fré06a], [Bur06]

- An abelian connected group A of finite Morley rank is indecomposable if it is not the sum of two proper definable subgroups. If $A \neq 1$, then A has a unique maximal proper definable connected subgroup $J(A)$, and if $A = 1$, let $J(1) = 1$.
- The reduced rank of any abelian indecomposable group A of finite Morley rank is $\bar{r}(A) = rk(A/J(A))$.
- For any group G of finite Morley rank and any positive integer r , we define

$$U_{0,r}(G) = \langle A \leq G \mid A \text{ is indecomposable definable abelian,} \\ \bar{r}(A) = r, A/J(A) \text{ is torsion-free} \rangle.$$

- A group G of finite Morley rank is said to be a $U_{0,r}$ -group whenever $G = U_{0,r}(G)$, and to be homogeneous if each definable connected subgroup of G is a $U_{0,r}$ -subgroup.
- The radical $U_0(G)$ is defined as follows. Set $\bar{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$ and set $U_0(G) = U_{0,\bar{r}_0(G)}(G)$.
- In any group G of finite Morley rank, a Sylow $U_{0,r}$ -subgroup is a maximal, definable, nilpotent $U_{0,r}$ -subgroup.
- In a group G of finite Morley rank, $U(G)$ is defined as the subgroup of G generated by its normal homogeneous $U_{0,s}$ -subgroups where s covers \mathbb{N}^* and by its normal definable connected subgroups of bounded exponent. A U -group is a group G of finite Morley rank such that $G = U(G)$.

The notion of reduced rank and the resulting unipotence theory, allowed a finer analysis of connected solvable groups in a way reminiscent of what torsion elements had allowed to achieve in such results as Facts 2.7, 2.8, 2.11, 2.20. Indeed, the first point of Fact 2.31 can be regarded as an analogue of Fact 2.20 while the points (6) and (7) refine Facts 2.7 and 2.8. The points (3), (4) and (5) are clear examples of nilpotent behaviour. It should also be emphasized that the “raison d’être” of the first two points is nothing but Fact 2.10.

Fact 2.31. –

- (1) [Bur04, Theorem 2.16] Let H be a connected solvable group of finite Morley rank. Then $U_0(H) \leq F(H)$.
- (2) [FJ05, Proposition 3.7] Let $G = NC$ be a group of finite Morley rank where N and C are nilpotent definable connected subgroups and N is normal in G . Assume that there is an integer $n \geq 1$ such that $N = \langle U_{0,s}(N) \mid 1 \leq s \leq n \rangle$ and $C = \langle U_{0,s}(C) \mid s \geq n \rangle$. Then G is nilpotent.
- (3) [Bur06, Lemma 2.3] Let G be a nilpotent group satisfying $U_{0,r}(G) \neq 1$. Then $U_{0,r}(Z(G)) \neq 1$.
- (4) [Bur06, Lemma 2.4] Let G be a nilpotent $U_{0,r}$ -group. If H is a definable proper subgroup of G then $U_{0,r}(N_G(H)/H) > 1$.

- (5) [Bur06, Theorem 2.9] *Let G be a nilpotent $U_{0,r}$ -group. Let $\{H_i | 1 \leq i \leq n\}$ be a family of definable subgroups such that $G = \langle \cup_i H_i \rangle$. Then $G = \langle U_{0,r}(H_i) | 1 \leq i \leq n \rangle$.*
- (6) [Bur06, Theorem 3.4] *Let G be a divisible nilpotent group of finite Morley rank, and let T be the torsion subgroup G . Then*

$$G = d(T) * U_{0,1}(G) * U_{0,2}(G) * \dots * U_{0,\text{rk}(G)}(G) .$$

- (7) [Bur06, Corollary 3.5] *Let G be a nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups D, B where D is divisible and B has bounded exponent. The latter group is connected if and only if G is connected.*

Let T be the torsion part of D . Then we have decompositions of D and B as follows.

$$D = d(T) * U_{0,1}(G) * U_{0,2}(G) \dots$$

$$B = U_2(G) \oplus U_3(G) \oplus \dots$$

For a prime p , $U_p(G)$ is the largest normal p -unipotent subgroup of G .

The work of the third author showed that the theory of unipotence is much better behaved when the unipotent groups in question are homogeneous in the sense of Definition 2.30. Remarkably, as points (1), (3) and (4) of Fact 2.32 illustrate, in order to find homogeneous groups it suffices to avoid central elements.

Fact 2.32. –

- (1) [Fré06a, Theorem 4.11] *Let G be a connected group of finite Morley rank. Assume that G acts definably by conjugation on H , a nilpotent $U_{0,r}$ -group. Then $[G, H]$ is a homogeneous $U_{0,r}$ -group.*
- (2) [Fré06a, Theorem 5.4] *Let G be a U -group. Then G has the following decomposition:*

$$G = B * U_{0,1}(G) * U_{0,2}(G) * \dots * U_{0,\bar{r}(G)}(G),$$

where

- (i) B is definable, connected, definably characteristic and of bounded exponent;
- (ii) $U_{0,s}(G)$ is a homogeneous $U_{0,s}$ -subgroup for each $s \in \{1, 2, \dots, \bar{r}(G)\}$;
- (iii) the intersections of the form $U_{0,s}(G) \cap U_{0,t}(G)$ are finite. In particular, if G does not contain a bad group, then

$$G = B \times U_{0,1}(G) \times U_{0,2}(G) \times \dots \times U_{0,\bar{r}(G)}(G).$$

- (3) [Fré06a, Corollary 6.8] *Let G be a solvable connected group of finite Morley rank. Then G' is a U -group.*
- (4) [Fré06a, Lemma 4.3] *Let G be a nilpotent $U_{0,r}$ -group. Then $G/Z(G)^\circ$ is a homogeneous $U_{0,r}$ -group.*

A natural question in this context was whether it was possible to develop a Sylow theory using the notions introduced in Definition 2.30. The second author's work answered this affirmatively in the context of connected solvable groups of finite Morley rank.

Fact 2.33. –

- (1) [Bur06, Lemma 6.2] *In a group G of finite Morley rank, the Sylow $U_{0,r}$ -subgroups are exactly those nilpotent $U_{0,r}$ -subgroups S such that $U_{0,r}(N_G(S)) = S$.*
- (2) [Bur06, Theorem 6.5] *Let H be a connected solvable group of finite Morley rank. Then the Sylow $U_{0,r}$ -subgroups of H are conjugate in H .*
- (3) [Bur06, Theorem 6.7] *Let H be a connected solvable group of finite Morley rank and let Q be a Carter subgroup of H . Then $U_{0,r}(H')U_{0,r}(Q)$ is a Sylow $U_{0,r}$ -subgroup of H , and every Sylow $U_{0,r}$ -subgroup has this form for some Carter subgroup of H .*
- (4) [Bur06, Corollary 6.9] *Let H be a connected solvable group of finite Morley rank and let S be a Sylow $U_{0,r}$ -subgroup of H . Then $N_H(S)$ contains a Carter subgroup of H .*

These results that we will use intensively in this paper have been key to the progress in local analysis in connected minimal simple groups of finite Morley rank. The facts below summarize the major ingredients of local analysis.

Fact 2.34. –

- (1) [Bur07, Lemma 2.1] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups satisfying $U_{p_1}(B_1) \neq 1$ and $U_{p_2}(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*
- (2) [Bur07, Corollary 2.2] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups of G . Then $F(B_1) \cap F(B_2)$ is torsion-free.*
- (3) [Bur07, Lemmas 3.5 and 3.6] *Let G be a minimal connected simple group. Assume that B_1 and B_2 are two distinct Borel subgroups such that $(B_1 \cap B_2)^\circ$ is maximal with respect to containment, and that $F(B_1) \cap F(B_2) \neq 1$. If $\bar{\tau}_0(B_1) \geq \bar{\tau}_0(B_2)$, then $\bar{\tau}_0(B_1) > \bar{\tau}_0(H)$ and $\bar{\tau}_0(B_2) = \bar{\tau}_0(H)$, where $H = (B_1 \cap B_2)^\circ$.*
- (4) [Bur07, Lemma 3.28] *We use the same notation as in the previous conclusion and let $X = F(B_1) \cap F(B_2)$. Then, $C_G^\circ(X)$ is non-nilpotent. Furthermore, if H is not abelian, then B_1 is the only Borel subgroup containing $C_G^\circ(X)$.*

Fact 2.35. – [Bur07, Proposition 4.1] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups of G . Let H be a definable connected subgroup of the intersection $B_1 \cap B_2$. Then the following hold:*

- (1) *H' is homogeneous or trivial.*
- (2) *Any definable connected nilpotent subgroup of $B_1 \cap B_2$ is abelian.*

Fact 2.36. – [Bur07, Theorem 4.3]

- (1) *Let G be a minimal connected simple, and let B_1, B_2 be two distinct Borel subgroups of G . Suppose that $H = (B_1 \cap B_2)^\circ$ is non-abelian. Then the following are equivalent:*
 - (a) *B_1 and B_2 are the only Borel subgroups G containing H .*
 - (i) *If B_3 and B_4 are distinct Borel subgroups of G containing H , then $(B_3 \cap B_4)^\circ = H$.*
 - (ii) *$C_G^\circ(H')$ is contained in B_1 or B_2 .*
 - (iii) *$\bar{\tau}_0(B_1) \neq \bar{\tau}_0(B_2)$.*

- (2) [Bur07, Lemma 3.28] *If one of the equivalent conditions of (1) holds and $\bar{r}_0(B_1) > \bar{r}_0(B_2)$, then B_1 is the only Borel subgroup containing $N_G(H')^\circ$.*
- (3) [Bur07, Consequence of Theorem 4.5 (4)] *If one of the equivalent conditions of (1) holds and $\bar{r}_0(B_1) > \bar{r}_0(B_2)$ and $r = \bar{r}_0(H')$, then $F_r(B_2)$ is non-abelian, where $F_r(X)$ denotes $U_{0,r}(F(X))$ with X a solvable connected group of finite Morley rank.*

We will finish our excursion on groups of finite Morley rank with a short overview of their geometric theory. Finite groups are discrete structures and their structure is best understood using counting arguments that frequently yield conjugacy theorems. On the other hand, density arguments tend to prevail in the realm of algebraic groups, and occasionally result in conjugacy results. In the theory of groups of finite Morley rank one has recourse to both resources, and occasionally profits from the interplay between the finite and the infinite. A nice example of such an interplay is provided by *Weyl groups*, a major theme of this article, of which the analysis will start in the next section.

The geometric analysis of groups of finite Morley rank mostly involves *genericity* arguments. Indeed, as was mentioned earlier in the context of tori, the more the nature of a generic element of a group of finite Morley rank is known, the better the group is understood. Certainly, one should not conclude from this remark that it suffices to understand the generic element of a group of finite Morley rank in order to understand the group fully. It is in fact a major question to what extent generic behaviour is also global. Nevertheless, in many cases, generic knowledge is very efficient.

A frequently encountered genericity notion is that of *generous* set since it allows to take into account the conjugates of a distinguished set under the action of the ambient group. A definable subset X of a group G of finite Morley rank is said to be *generous in G* (or shortly, “generous” in case the ambient group is clear) if the union of its conjugates is generic in G . This notion was introduced and studied in [Jal06]. The following were proven in [Jal06]:

Fact 2.37. – *Let G be a group of finite Morley rank and H a definable, generous subgroup of G .*

- (1) [Jal06, Lemma 2.2] *The subgroup H is of finite index in $N_G(H)$.*
- (2) [Jal06, Lemma 2.3] *If X is a definable subset of H that is generous in G , then X is generous in H .*
- (3) [Jal06, Lemma 2.4] *If H is connected and X is a definable generic subset of H , then X is generous in G .*

The first point in the above fact that, despite its simple nature and proof, gives immediately a clear idea about the relationship between generic sets and Weyl groups.

The following characterization of generosity is due to Cherlin who was inspired by [Jal06]. It is a relatively simple but efficient illustration of the geometry of genericity arguments.

Fact 2.38. – [ABC08, Lemma IV 1.25][Jal06, Section 3.2] *Let G be a connected group of finite Morley rank and H definable, connected, and almost self-normalizing subgroup of G . Let \mathcal{F} be the family of all conjugates of H in G . Then the following are equivalent.*

- (1) *H is generous in G .*

(2) *The definable set*

$$H_0 = \{h \in H : \{X \in \mathcal{F} : h \in X\} \text{ is finite}\}$$

is generic in H .

(3) *The definable set*

$$G_0 = \{x \in \bigcup_{g \in G} H^g : \{X \in \mathcal{F} : x \in X\} \text{ is finite}\}$$

is generic in G .

As we mentioned at the beginning of our discussion of genericity as well as before Fact 2.23, there is a close connection between conjugacy and genericity although this does not in general necessitate an implication in either direction. Indeed, the conjugacy results on decent and pseudo-tori go through genericity arguments. In [Jal06], Jaligot proved the conjugacy of generous Carter subgroups of groups of finite Morley rank, while this is a major open problem in general. The only known answer that does not depend on the generosity assumption is for minimal connected simple groups in [Fr  08], and even under the strong assumption of minimality, the lack of a clear description of a generic element complicated the proofs considerably.

Fact 2.39. – [Jal06, Part of Corollary 3.8] *Let G be a group of finite Morley rank and C a Carter subgroup of G . Then the following are equivalent:*

- (1) *C is generous in G .*
- (2) *C is generically disjoint from its conjugates.*

A definable set X is *generically disjoint* from its conjugates if $\text{RM}(X \setminus \bigcup_{g \in G \setminus \text{Stab}_G(X)} X^g) = \text{RM}(X)$.

Fact 2.40. – *Let G be a group of finite Morley rank. Then the following conditions hold:*

- (1) [Jal06, Theorem 3.1] *its generous Carter subgroups are conjugate;*
- (2) [CJ04, Lemma 3.5] [FJ08, Theorem 3.11] *if G is solvable, its Carter subgroups are generically disjoint and generous.*

We finish this section stating an observation about minimal connected simple groups that in particular illustrate the connection between genericity, Carter subgroups and torsion elements.

Fact 2.41. – [AB09, Proposition 3.6] *Let G be minimal connected simple group. Then*

- (1) *either G does not have torsion,*
- (2) *or G has a generous Carter subgroup.*

In the next section, we will start seeing in action the connection between finite and generic in the analysis of Weyl groups.

3. THE WEYL GROUP OF A GROUP OF A MINIMAL CONNECTED SIMPLE GROUP OF FINITE MORLEY RANK

There are several definitions proposed for Weyl groups in groups of finite Morley rank: in a group G of finite Morley rank, one can propose $N_G(C)/C$ where C is a Carter subgroup of G , or $N_G(T)/C_G(T)$ where T is a maximal decent or pseudo-torus. In simple algebraic groups, these possibilities yield natural, uniquely defined,

robust notions that are also equivalent. This is not known in an arbitrary simple group of finite Morley rank.

Fact 2.25 (1), or the definition of a decent or pseudo-torus show that one can define a notion of Weyl group in a group of finite Morley rank. Nevertheless, the definition using Carter subgroups cannot yield a uniquely defined notion as long as it is not known whether in general, Carter subgroups are conjugate in groups of finite Morley rank, an open problem. On the other hand, thanks to Fact 2.23, this problem is overcome in the case of the definitions involving tori. Motivated by this fact, we define the Weyl group $W(G)$ of a group G of finite Morley rank to be $N_G(T)/C_G(T)$ where T is any maximal decent torus of G .

Our first target in the present section is to verify that in a minimal connected simple group, the definition of a Weyl group that we have adopted is in fact equivalent to the other above-mentioned possibilities. This will be done mainly in Proposition 3.2 and followed up in Corollaries 3.4 and 3.5.

The second target of this section is to use our development of a robust notion of Weyl group in the analysis of another well-known property of simple algebraic groups ([Hum81, Theorem 23.1]) in the context of groups of finite Morley rank, namely the self-normalization of Borel subgroups. This problem is open even in the context of minimal connected simple groups of finite Morley rank. We will prove in Theorem 3.13 that the property holds in a minimal connected simple group under additional hypotheses.

An important ingredient of our arguments is the conjugacy of Carter subgroups in minimal connected simple groups (Fact 2.25 (4)). We will also need the following fact which can be regarded as a very weak form of self-normalization:

Fact 3.1. – [AB09, Lemma 4.3] *If B is a Borel subgroup of a minimal connected simple group G such that $U_p(B) \neq 1$ for some prime number p , then p does not divide $[N_G(B) : B]$.*

Proposition 3.2. – *Let G be a minimal connected simple group, and let C be a Carter subgroup of G . Then the Weyl group $W(G)$ of G is isomorphic to $N_G(C)/C$.*

PROOF – Let T be a maximal decent torus of G . Then T is contained in a Carter subgroup of G (Fact 2.25 (2)) and, by the conjugacy of Carter subgroups (Fact 2.25 (4)), we may assume $T \leq C$. By Fact 2.7, we have $C \leq C_G(T)^\circ$. If T is non-trivial, then $C_G(T)$ is a connected solvable subgroup of G by Fact 2.24 (3). In particular C is self-normalizing in $C_G(T)$ (Fact 2.25 (5)), and Fact 2.25 (3) and a Frattini Argument yield $N_G(T) = C_G(T)N_G(C)$. Hence we obtain

$$N_G(C)/C \simeq N_G(T)/C_G(T) \simeq W(G),$$

and we may assume $T = 1$. By Fact 2.23 (1), there is no non-trivial decent torus in G .

We assume toward a contradiction that $N_G(C)/C$ is non-trivial. Then there is a prime p dividing the order of $N_G(C)/C$. Let S be a Sylow p -subgroup of G . By the previous paragraph and by Fact 2.14 (2), S° is a p -unipotent subgroup of G . Moreover, it is non-trivial by Fact 2.16. Let B be a Borel subgroup containing S° . Then we have $S^\circ \leq U_p(B)$ and Fact 2.34 (1) shows that B is the unique Borel subgroup containing S° . In particular, S normalizes B and $U_p(B)$, and we obtain $S^\circ = U_p(B)$ by maximality of S . Thus we have $N_G(B) = N_G(S^\circ)$.

Let D be a Carter subgroup of B (Fact 2.25 (1)). If a B -minimal section \bar{A} of S° is not centralized by B , then $B/C_B(\bar{A})$ is definably isomorphic to a definable subgroup

of K^* for a definable algebraically closed field K of characteristic p (Fact 2.10), and Fact 2.22 shows that $B/C_B(\overline{A})$ is a decent torus. Then there is a non-trivial decent torus in B by Fact 2.24 (1), contradicting our first paragraph. Thus D centralizes each B -minimal section of S° , and this implies $S^\circ \leq D$ since $N_B(D)^\circ = D$. Now $N_G(D)$ normalizes $S^\circ = U_p(D)$ and we have $N_G(D) \leq N_G(S^\circ) = N_G(B)$. In particular D is a Carter subgroup of G and we may assume $D = C$ (Fact 2.25 (4)). By the conjugacy of Carter subgroups in B (Fact 2.25 (3)) and a Frattini Argument, we obtain $N_G(B) = BN_G(C)$ and

$$N_G(C)/C = N_G(C)/(N_G(C) \cap B) \simeq N_G(B)/B.$$

This implies that p divides the order of $N_G(B)/B$, contradicting Fact 3.1. \square

This result has the following consequence, which is similar to a classical result for algebraic groups [Hum81, Exercise 6 p.142].

Corollary 3.3. – *If C is a Carter subgroup of a minimal connected simple group G , then C is a maximal nilpotent subgroup.*

PROOF – Let D be a nilpotent subgroup of G containing C . By Fact 2.2, we may assume D is definable. Since $N_G(C)/C$ is finite, Fact 2.9 (1) implies that C has finite index in D , and thus $C = D^\circ$. Let T be the maximal decent torus of C . Then T is maximal in G by Fact 2.25 (2) and (4). Thus, if $T = 1$, then Proposition 3.2 gives $N_G(C) = C$ and $D = C$. On the other hand, if $T \neq 1$, then $C_G(T)$ is a connected solvable group by Fact 2.24 (3), and it contains D (Fact 2.7). Now, by Fact 2.25 (5), we obtain $D \leq N_{C_G(T)}(C) = C$, proving the maximality of C . \square

Corollary 3.4. – *Let G be a minimal connected simple group, and let S be a non-trivial p -torus for a prime p . Then $N_G(S)/C_G(S)$ is isomorphic to a subgroup of $W(G)$. Moreover, if S is maximal, then we have $N_G(S)/C_G(S) \simeq W(G)$.*

PROOF – By Fact 2.25 (2), S is contained in a Carter subgroup C of G . By Fact 2.7, we have $C \leq C_G(S)^\circ$. Since S is non-trivial, then $C_G(S)$ is a connected solvable group Fact 2.24 (3), and C is self-normalizing in $C_G(S)$ (Fact 2.25 (5)). Now a Frattini Argument yields $N_G(S) = C_G(S)N_{N_G(S)}(C)$, and $N_G(S)/C_G(S)$ is isomorphic to a subgroup of $N_G(C)/C \simeq W(G)$.

Moreover, if S is maximal, then S is characteristic in C , and we have $N_G(C) = N_{N_G(S)}(C)$. Hence we obtain $W(G) \simeq N_G(C)/C \simeq N_G(S)/C_G(S)$. \square

Corollary 3.5. – *Let G be a minimal connected simple group, and let T be a maximal pseudo-torus of G . Then $W(G)$ is isomorphic to $N_G(T)/C_G(T)$.*

PROOF – We proceed as in the first paragraph of the proof of Proposition 3.2. By Facts 2.25 (2) and 2.24 (2), T is a central subgroup of a Carter subgroup C of G . If T is non-trivial, then Fact 2.24 (4) and Fact 2.25 (3) and (5) provide $W(G) \simeq N_G(C)/C \simeq N_G(T)/C_G(T)$. It then follow from Proposition 3.2 that $W(G) \simeq N_G(C)/C$, so we may assume $T = 1$.

Let S be a maximal decent torus of G . By Fact 2.23 (2), S is conjugate with a subgroup of T , so $S = T = 1$ and we obtain the result. \square

Now, we move on to the problem of self-normalization of Borel subgroups. We will need several results from [Del08] and [BD09]. We will thus carry out an active survey of these papers in order to extract from them our needs in a form that is more suitable for us and not necessarily available in these two sources.

We begin by reformulating a large portion of the main theorem of [Del08]. The use of the conjugacy of Carter subgroups of minimal connected simple groups and the results of last section provide the missing uniformity in the statement of [Del08, Théorème-Synthèse]. In fact, this is our sole contribution. For the sake of completeness, we detail how these new ingredients intervene in the proof together with Fact 2.24 (3).

Fact 3.6. – (Particular case of [Del08, Théorème-Synthèse]) *Let G be a minimal simple group of odd type with non-trivial Weyl group $W(G)$. Then G satisfies one of the following three conditions:*

- $G \simeq \mathrm{PSL}_2(K)$ for an algebraically closed field K of characteristic $p \neq 2$;
- $|W(G)| = 2$, the Prüfer 2-rank of G is one, the involutions of G are conjugate, and G has an abelian Borel subgroup C such that $N_G(C) = C \rtimes \langle i \rangle$ where i is an involution inverting C ;
- $|W(G)| = 3$, the Prüfer 2-rank of G is two, and the Carter subgroups of G are abelian and divisible, but they are not Borel subgroups.

Furthermore, each Carter subgroup of G has the form $C_G(T)$ for a 2-torus T of G .

PROOF – First we assume that the Prüfer 2-rank of G is one. Then, by [Del08, Théorème-Synthèse], either we have $G \simeq \mathrm{PSL}_2(K)$ for an algebraically closed field K of characteristic $p \neq 2$, or $|W(G)| = 2$ and G has an abelian Borel subgroup C such that $N_G(C) = C \rtimes \langle i \rangle$ for an involution i inverting C . In the first case, the Carter subgroups are maximal tori. In particular they are of the form $C_G(T)$ for a 2-torus T of G . In the second case, let T be a maximal 2-torus of G . Then T is in the centre of a Carter subgroup C of G (Fact 2.25 (2) and Fact 2.7). But, by Fact 2.25 (4), each Carter subgroup of G is a Borel subgroup. Hence we have $C = C_G(T)^\circ$, and the result follows from the connectedness of $C_G(T)$ (Fact 2.24 (3)) when the Prüfer 2-rank of G is one.

Now we may assume that the Prüfer 2-rank of G is two. Note that, for this case, it is not clear that the group W corresponds in $W(G)$ in [Del08]. Let S be a Sylow 2-subgroup of G . First we show that $|W(G)| = 3$. By Corollary 3.4, we have $W(G) \simeq N_G(S^\circ)/C_G(S^\circ)$. On the other hand $C_G(S^\circ)$ is connected by Fact 2.24 (3), and S° is characteristic in $C_G(S^\circ)$. Hence we obtain $W(G) \simeq N_G(C_G(S^\circ)^\circ)/C_G(S^\circ)^\circ$ and $|W(G)| = 3$.

Secondly we prove that the Carter subgroups of G are abelian and divisible, and of the form $C_G(T)$ for a 2-torus T . Since G is of odd type, S° is a non-trivial 2-torus, and it is contained in a Carter subgroup C of G (Fact 2.25 (2)). Moreover, S° is central in C Fact 2.7. Since [Del08, Théorème-Synthèse] says that $C_G(S^\circ)^\circ$ is abelian and divisible, we obtain $C = C_G(S^\circ)^\circ$. Moreover, C is not a Borel subgroup by [Del08, Théorème-Synthèse]. Now the result follows from Fact 2.25 (4) and from the connectedness of $C_G(S^\circ)$ (Fact 2.24 (3)). \square

As for [BD09], the second part of the following fact, rather than the cyclicity of the Weyl group, will be needed in the sequel.

Fact 3.7. – [BD09, Theorem 4.1] *Let G be a minimal connected simple group, T a maximal decent torus of G , and τ the set of primes p such that \mathbb{Z}_{p^∞} embeds into T . Then $W(G)$ is cyclic, and has an isomorphic lifting to G . Moreover, no element of τ divides $|W(G)|$, except possibly 2.*

Note that the results of [BD09, §3] do not need that the group G be degenerate, but just that $|W(G)|$ be odd. This increases their relevance for us in conjunction with results from [BC08b]. In particular, the following fact holds. We will denote τ' the complementary set of prime numbers when τ is a subset of prime numbers.

Fact 3.8. – [BD09, §3][BC08b, §5] *Let G be a minimal connected simple group, T a maximal decent torus of G , and τ the set of primes p such that \mathbb{Z}_{p^∞} embeds into T . If $W(G)$ is non-trivial and of odd order, then the following conditions hold:*

- (1) [BC08b, Corollary 5.3] *the minimal prime divisor of $|W(G)|$ does not belong to τ ;*
- (2) *if a is a τ' -element of $N_G(T)$, then $C_{C_G(T)}(a)$ is trivial;*
- (3) *$C_G(T)$ is a Carter subgroup of G ;*
- (4) *if B_T is a Borel subgroup containing $C_G(T)$, and if either there is a τ' -element normalizing T and B_T , or there is a prime q such that the Prüfer q -rank of T is ≥ 3 , then $C_G(T) = B_T$.*

It is worth noting that point (4) of Fact 3.8 is true even when $W(G)$ is of even order.

The following strengthens Fact 3.8 (3)

Corollary 3.9. – *Let G be a minimal connected simple group, and T a maximal decent torus of G . If $W(G)$ is non-trivial, then $C_G(T)$ is a Carter subgroup of G and any Carter subgroup of G has this form.*

PROOF – By Fact 2.25 (4), we have just to prove that $C_G(T)$ is a Carter subgroup of G . By Fact 3.8 (3), we may assume that $W(G)$ is of even order. By Fact 2.16 and the classification of simple groups of even type, either we have $G \simeq \text{PSL}_2(K)$ for an algebraically closed field K , or G is of odd type. Hence we may assume that G is of odd type.

By Fact 2.25 (2), there is a Carter subgroup C of G containing T , and T is central in C by Fact 2.7. But Fact 3.6 shows that $C = C_G(S)$ for a 2-torus S of G . Thus, T contains S by maximality of T (Fact 2.7), and $C = C_G(T)$. \square

Lemma 3.10. – *Let B_1 and B_2 be two generous Borel subgroups of a minimal connected simple group G . Then there exists $g \in G$ such that $B_1 \cap B_2^g$ contains a generous Carter subgroup of G .*

PROOF – Let C_i be a Carter subgroup of B_i for $i = 1, 2$. Then C_i is generous in B_i by Fact 2.40 (2), and C_i is generous in G by Fact 2.37 (3). This implies that C_i has finite index in its normalizer in G (Fact 2.37 (1)), therefore C_i is a Carter subgroup of G . Now C_1 and C_2 are conjugate (Fact 2.40 (1)), and there exists $g \in G$ such that $C_2^g = C_1 \leq B_1 \cap B_2^g$. \square

Lemma 3.11. – *Let G be a minimal connected simple group with a nilpotent Borel subgroup B . Then B is a Carter subgroup of G , and the generous Borel subgroups of G are conjugate with B , and they are generically disjoint.*

PROOF – By Fact 2.5, B is non-trivial, so $N_G(B)^\circ$ is solvable and B is a Carter subgroup of G .

Let B_0 be a generous Borel subgroup of G . An application of Fact 2.37 (3) and (1) shows that the Carter subgroups of B_0 are also Carter subgroups of G . Then

Fact 2.25 (4) implies that B is conjugate to a Carter subgroup of B_0 , and thus to B_0 . The generic disjointness follows from Fact 2.39. \square

Now, we will prove the self-normalization theorem. In the end of the proof, as noted there as well, we could quote Fact 3.7 to finish quickly. Nevertheless, we prefer to give a slightly longer but direct argument for two reasons. The first is that the quick ending is in fact longer in that it uses the full force of [BD09], which we do not need here. The second and more important reason is that in Section 11, it will be crucial to have a clean self-normalization argument that deals with the special case when the Weyl group is of odd order in order to avoid referring to [Del08]. Fact 3.8 makes it possible to achieve this goal. Moreover, as was noted before Fact 3.8, the validity of this fact is not restricted to groups of degenerate type.

The direct approach will use the following classical result:

Fact 3.12. – [ABC08, Lemmas IV.10.16 and IV.10.18] *Let T be a p -torus of Prüfer p -rank 1 or 2, where p is a prime, and α an automorphism of T of order p , with a finite centralizer in T . Then $p \in \{2, 3\}$.*

Theorem 3.13. – *Any non-nilpotent generous Borel subgroup B of a minimal connected simple group G is self-normalizing.*

PROOF – We consider a non-nilpotent generous Borel subgroup B of a minimal connected simple group G . If $|W(G)|$ is even, then Fact 2.16 and the classification of simple groups of even type shows that either B is self-normalizing, or G is of odd type. In the second case, Fact 3.6 and Lemma 3.11 imply that $G \simeq \text{PSL}_2(K)$ for an algebraically closed field K , so B is self-normalizing. Hence we may assume that $|W(G)|$ is odd.

We assume toward a contradiction that B is not self-normalizing. By Lemma 3.10, B contains a (generous) Carter subgroup C of G . By Fact 2.25 (3) and a Frattini argument, we have $N_G(B) = BN_{N_G(B)}(C)$, so C is not self-normalizing, and the Weyl group of G is non-trivial (Proposition 3.2). Moreover $|N_G(B)/B|$ divides $|W(G)|$. Let T be the maximal decent torus of C . By Corollary 3.9, T is a maximal decent torus of G and we have $C = C_G(T)$.

Let p be a prime divisor of $|N_G(B)/B|$. Since we have $N_G(B) = BN_{N_G(B)}(C)$, there is a p -element w in $N_{N_G(B)}(C) \setminus B$ such that $w^p \in B$. In particular we have $w \in N_G(T) \setminus C_G(T)$ and, by Fact 3.8 (4), the maximal p -torus R of T is non-trivial of Prüfer p -rank 1 or 2.

At this point, Fact 3.7 allows to finish the proof since it yields a contradiction. As was explained above, we will not do this and give a more direct final argument.

Let $R_0 = C_R(w)^\circ$. It is a p -torus and we have $w \in C_G(R_0)$. Moreover, we have $C \leq C_G(R) \leq C_G(R_0)$ and $C_G(R_0)$ is connected by Fact 2.24 (3). Thus, if R_0 is non-trivial, then C is a Carter subgroup of the connected solvable subgroup $C_G(R_0)$, and Fact 2.25 (5) yields $w \in N_{C_G(R_0)}(C) = C$, contradicting our choice of w . Hence R_0 is trivial and, since Fact 2.25 (5) implies $w^p \in N_B(C) = C = C_G(T) \leq C_G(R)$, the element w induces an automorphism φ of order p of R such that $C_R(\varphi)$ is finite. Then, since we have $p \neq 2$, Fact 3.12 implies $p = 3$. But $|W(G)|$ is odd, so p is the smallest prime divisor of $|W(G)|$, contradicting Fact 3.8 (1). The proof is over. \square

4. TETRACHOTOMY THEOREM

This section sets the main lines for the rest of this article except for the final section that is directly related to Section 3, and the discussion of reducts in Section 10. In Theorem 4.1, we will carry out a fine analysis of minimal connected simple groups according to two criteria. The first criterion is the existence of a non-trivial Weyl group. This criterion is motivated by the important role played by Weyl groups in minimal connected simple groups of finite Morley rank. When the Weyl group is non-trivial, it determines many structural aspects of the ambient group as was exemplified in the classification of simple groups of even type or in [Del08]. On the other hand, when it is trivial, the ambient group has very high chances of being torsion-free, and the arguments tend to use the geometry of G as in [Fr  08].

Our second criterion is the size of the intersections of Borel subgroups. It was already noticed in [Jal01] that the lack of intersection between Borel subgroups makes it very difficult to analysis minimal connected simple groups. “Large” intersections, like in the classification of the finite simples groups, allow a certain kind of local analysis. We have set the following concrete criterion in order to measure whether a minimal connected simple group admits largely intersecting Borel subgroups: the absence of a Borel subgroup generically disjoint from its conjugates other than itself.

The following table introduces the four types of groups that emerge from these two criteria:

		A Borel subgroup generically disjoint from its conjugates	
		exists	does not exist
Weyl group	trivial	(1)	(2)
	non-trivial	(3)	(4)

Theorem 4.1. – (Tetrachotomy theorem) *Any minimal connected simple group G satisfies exactly one of the following four conditions:*

- G is of type (1), its Carter subgroups are generous and any generous Borel subgroup is generically disjoint from its conjugates;
- G is of type (2), it is torsion-free and it has neither a generous Carter subgroup, nor a generous Borel subgroup;
- G is of type (3), its Carter subgroups are generous, and they are generous Borel subgroups;
- G is of type (4), and its Carter subgroups are generous.

In the sequel, by “type (i)” we will mean one of the four types characterized in Theorem 4.1.

Remark 4.2. –

- *Bad groups* [BN94, Chapter 13], and more generally *full Frobenius groups* [Jal01], are examples of groups of type (1). The existence of any of these groups is a well-known open problem.
- The minimal connected simple groups with a nongenerous Carter subgroup are of type (2) and are analyzed in [Fr  08].
- By Lemma 3.11, any minimal connected simple group with a nilpotent Borel subgroup is of type (1) or (3).
- The group $\mathrm{PSL}_2(K)$ for an algebraically closed field K , is of type (4).

- By Fact 2.16, the classification of simple groups of even type, and Fact 3.6, a minimal connected simple group with an involution is of one of the types (1), (3) or (4).

The following lemma seems to be of general interest.

Lemma 4.3. – *Let G be a minimal connected simple group with trivial Weyl group. Then every Carter subgroup of G is contained in a unique Borel subgroup of G .*

PROOF – Let C be a Carter subgroup of G . Since $W(G)$ is trivial, C is self-normalizing in G (Proposition 3.2). If there is $g \notin B$ such that B^g contains C , then there is $u \in B^g$ such that $C^{gu} = C$ (Fact 2.25 (3)). In particular we have $gu \in N_G(C) = C \leq B^g$ and $g \in B^g$. This implies $g \in B$, contradicting our choice of g . Hence, for each $g \notin B$, we have $C \not\leq B^g$, so each Carter subgroup of G is contained in a unique conjugate of B . \square

PROOF OF THEOREM 4.1 – First we note that, by Fact 2.25 (4), either all the Carter subgroups of G are generous, or G has no generous Carter subgroup. We will divide our discussion into two cases:

Case I: $W(G) = 1$, equivalently, G is of type (1) or (2).

We first show that any generous Borel subgroup of G is generically disjoint from its conjugates. Let B be a generous Borel subgroup of G , and let C be a Carter subgroup of B . By Facts 2.40 (2) and 2.37 (3), the Carter subgroup C of B is generous in G , and it is a Carter subgroup of G by Fact 2.37 (1).

In order to simplify notation, let us set $\mathcal{C}_G = \{C^g | g \notin N_G(B)\}$ and $\mathcal{C}_B = \{C^b | b \in B\}$. By Lemma 4.3 and Fact 2.25 (3), these two sets form a disjoint union equal to the entire set of conjugates of C in G , equivalently (Fact 2.25 (4)) to the entire set of Carter subgroups of G . By Lemma 4.3, $C \setminus \bigcup \mathcal{C}_G \supseteq C \setminus \left(\bigcup_{g \notin N_G(C)} C^g\right)$. It then follows from Fact 2.39 that $C \setminus \bigcup \mathcal{C}_G$ is generic in C .

Now we assume toward a contradiction that B is not generically disjoint from its conjugates, equivalently the set $B \cap \left(\bigcup_{g \notin N_G(B)} B^g\right)$ is generic in B . Then, since $\bigcup_{g \notin N_G(B)} B^g$ is invariant under the action of B by conjugation, the generosity of C in B (Fact 2.40 (2)) implies the one of $C \cap \left(\bigcup_{g \notin N_G(B)} B^g\right)$ in B . Thus $C \cap \left(\bigcup_{g \notin N_G(B)} B^g\right)$ is generic in C by Fact 2.37 (2).

We consider $X := B \setminus \bigcup \mathcal{C}_B$. Then we have $B = X \cup \left(\bigcup \mathcal{C}_B\right)$, so we find

$$\begin{aligned} \bigcup_{g \notin N_G(B)} B^g &= \bigcup_{g \notin N_G(B)} (X \cup \bigcup \mathcal{C}_B)^g \\ &= \bigcup_{g \notin N_G(B)} X^g \cup \bigcup_{g \notin N_G(B)} \left(\bigcup \mathcal{C}_B\right)^g \\ &= \bigcup_{g \notin N_G(B)} X^g \cup \bigcup \mathcal{C}_G, \end{aligned}$$

and we obtain

$$C \cap \bigcup_{g \notin N_G(B)} B^g = (C \cap \bigcup_{g \notin N_G(B)} X^g) \cup (C \cap \bigcup \mathcal{C}_G).$$

But $C \setminus \bigcup \mathcal{C}_G$ is generic in C , therefore $C \cap \bigcup \mathcal{C}_G$ is not generic in C , and since the previous paragraph says that $C \cap \left(\bigcup_{g \notin N_G(B)} B^g\right)$ is generic in C too, we find the genericity of $C \cap \bigcup_{g \notin N_G(B)} X^g$ in C . Consequently, since C is generous in G , the set X is generous in G by Fact 2.37 (3). Now the set X is generous in B by Fact 2.37 (2), and since it is invariant under the action of B by conjugation, it is generic in B . This contradicts that C is generous in B (Fact 2.40 (2)) and proves that B is generically disjoint from its conjugates in G .

If G is of type (1), it remains to prove that G has a generous Carter subgroup. But a Borel subgroup B of G , generically disjoint from its conjugates, is generous in G by Fact 2.38. Hence G has a generous Carter subgroup Lemma 3.10.

If G is of type (2), the argument above shows that G has no generous Borel subgroup. In particular, G has no generous Carter subgroup, and G is torsion-free by Fact 2.41.

Case II: G is of type (3) or (4).

In this case, G is not torsion-free, and its Carter subgroups are generous by Fact 2.41.

If G is of type (3), we show that its Carter subgroups are generous Borel subgroups. Let B be a Borel subgroup of G generically disjoint from its conjugates. Then B is generous in G by Fact 2.38, and B contains a generous Carter subgroup C by Lemma 3.10. Since $W(G)$ is non-trivial, there exists $w \in N_G(C) \setminus C$ (Proposition 3.2), and Fact 2.25 (5) implies $w \notin B$. If $B \neq C$, then Theorem 3.13 gives $B^w \neq B$ and $C \leq B^w$. This implies that B is generically covered by its conjugates (Fact 2.40 (2)), and contradicts our choice of B . Hence we have $B = C$, and any Carter subgroup of G is a generous Borel subgroup. Conversely, any generous Borel subgroup of G is a Carter subgroup of G by Lemma 3.11. \square

5. STRUCTURE OF CARTER SUBGROUPS IN SIMPLE GROUPS OF TYPE (4)

As Remark 4.2 suggests, minimal connected simple algebraic groups over algebraically closed fields are of type (4). Thus, one expects simple groups of type (4) to have properties close to those of algebraic groups. The main result of this section, Theorem 5.2, provides evidence in this direction by sharpening the following result, which, together with Lemma 3.11, implies that the Prüfer p -rank of a minimal connected simple group of type (4) and of degenerate type is bounded by 2 for any prime p .

Fact 5.1. – [BD09, Theorem 3.1] *Let G be a minimal connected simple group of degenerate type. Suppose also that G has a non-trivial Weyl group $W = N_G(T)/C_G(T)$ where T is maximal decent torus. Then the Cartan subgroup $C_G(T)$ is nilpotent, and thus is a Carter subgroup of G .*

Moreover, $C_G(T)$ is actually a Borel subgroup if either $C_G(T)$ is not abelian or G has Prüfer q -rank at least 3 for some prime q .

Theorem 5.2. – *Let G be a minimal connected simple group of type (4). Then there is an interpretable field K such that each Carter subgroup definably embeds in $K^* \times K^*$.*

PROOF – Let C be a Carter subgroup of G , and let B be a Borel subgroup containing C such that either $U_q(B)$ is non-trivial for a prime q , or the integer $r = \bar{r}_0(B)$ is maximal for any such Borel subgroup. By Lemma 3.11, B is non-nilpotent and we have $B \neq C$. Since $W(G)$ is non-trivial, Proposition 3.2 gives $N_G(C) \neq C$, and Fact 2.25 (5) shows that B does not contain $N_G(C)$. Hence by Theorem 3.13, there exists $w \in N_G(C) \setminus N_G(B)$. In particular, $C \leq B \cap B^w$ is abelian by Fact 2.35 (2).

If we have $U_q(B) \neq 1$ for a prime q , then B has a G -minimal subgroup A of exponent q . If $C_C(A, A^w)$ is non-trivial, then $C_G(C_C(A, A^w))^\circ$ is a proper connected definable subgroup of G containing A and A^w . Hence Fact 2.34 (1) shows that B (resp. B^w) is the unique Borel subgroup of G containing $C_G(C_C(A, A^w))^\circ$.

This contradicts $B \neq B^w$. Consequently $C_C(A, A^w)$ is trivial, and C is definably isomorphic to a subgroup of $B/C_B(A) \times B^w/C_{B^w}(A^w) \simeq B/C_B(A) \times B/C_B(A)$. If $B/C_B(A) = 1$, then C is trivial, contradicting that C is a Carter subgroup of G . So, by Fact 2.10, there is an interpretable field K of characteristic q such that $B/C_B(A)$ is definably isomorphic to a subgroup of K^* . Hence we may assume that $U_q(B) = 1$ for each prime q , so $r > 0$.

We show that B has a B -minimal homogeneous $U_{0,r}$ -subgroup. By Fact 2.31 (1), $U_0(B)$ is nilpotent, and by Fact 2.32 (1) $[B, U_0(B)]$ is a homogeneous $U_{0,r}$ -subgroup of B , so we may assume $[B, U_0(B)] = 1$. Then $U_0(B)$ is central in B , and C contains $U_0(B)$, so we have $U_0(C) = U_0(B) \leq Z(B)$. Thus we obtain

$$U_0(B^w) = U_0(B)^w = U_0(C)^w = U_0(C) = U_0(B),$$

and $U_0(B)$ is central in $\langle B, B^w \rangle = G$, contradicting the simplicity of G . Hence B has a B -minimal homogeneous $U_{0,r}$ -subgroup A .

If $C_C(A, A^w) = 1$, then C is definably isomorphic to a subgroup of $B/C_B(A) \times B^w/C_{B^w}(A^w) \simeq B/C_B(A) \times B/C_B(A)$. If $B/C_B(A) = 1$, then $C = 1$, which contradicts that C is a Carter subgroup of G . So, by Fact 2.10, there is an interpretable field K such that $B/C_B(A)$ is definably isomorphic to a subgroup of K^* . Hence we may assume that $C_C(A, A^w)$ is non-trivial.

Let B_0 be a Borel subgroup of G containing $N_G(C_C(A, A^w))^\circ$. Then B_0 contains C , A and A^w . Since $B \neq B^w$, we have either $B_0 \neq B$ or $B_0 \neq B^w$. In the first case we consider $H_0 = (B_0 \cap B)^\circ$ and, in the second case, $H_0 = (B_0 \cap B^w)^\circ$. In particular H_0 contains C , and either A or A^w , so we have $\bar{\tau}_0(H_0) = r$.

Let B_1 and B_2 be two distinct Borel subgroups of G containing H_0 , such that $H = (B_1 \cap B_2)^\circ$ is maximal among all the choices of distinct Borel subgroups B_1 and B_2 . Since B_1 and B_2 contain H_0 , they contain C , and they are generous in G . Since $\bar{\tau}_0(H_0) = r$, the maximality of r yields $\bar{\tau}_0(H) = \bar{\tau}_0(B_1) = \bar{\tau}_0(B_2) = r$. Thus, by Fact 2.31 (1), $U_0(H) \leq F(B_1) \cap F(B_2)$. In particular, this intersection is non-trivial. But Fact 2.34 (3) implies that $F(B_1) \cap F(B_2) = 1$, a contradiction. \square

6. LOCAL ANALYSIS AND CARTER SUBGROUPS

In this section, we will use local analytic methods to refine our understanding of the relationships between Carter subgroups of minimal connected simple groups, the Borel subgroups containing these and the Weyl group of the ambient group. The conclusions, that will provide tools for the sequel, are of independent interest. They use such sources as [Bur06], [Bur07], [Fré06a].

Proposition 6.1. – *Let H be a subgroup of a minimal connected simple group G . If H contains a Carter subgroup C of G , then H is definable, and either it is contained in $N_G(C)$, or it is connected and self-normalizing.*

PROOF – We may assume that H is not contained in $N_G(C)$, and that H is proper in G . First we show that H is definable. The subgroup H_0 generated by the conjugates of C contained in H is definable and connected by Fact 2.3. In particular H_0 is solvable and, by conjugacy of Carter subgroups in H_0 (Fact 2.25 (3)) and a Frattini argument, we obtain $H = H_0 N_H(C)$. Thus H_0 has finite index in H , so H is definable. Note also that $H^\circ = H_0$.

We note that the condition $H \not\leq N_G(C)$ implies $H^\circ \not\leq N_G(C)$. Thus, if the result holds for connected groups, then H° is self-normalizing, and we have $H = H^\circ$. Hence we may assume that H is connected. In particular, H is solvable.

We assume toward a contradiction that H is a maximal connected counterexample to the proposition. Since the conjugacy of Carter subgroups in H (Fact 2.25 (3)) and a Frattini argument yield $N_G(H) = HN_{N_G(H)}(C)$, the quotient group $N_G(H)/H$ is isomorphic to a subgroup of $W(G)$ by Proposition 3.2, and $W(G)$ is non-trivial. By Theorem 4.1, C is generous in G .

We consider a Borel subgroup B containing H . By Fact 2.25 (5), the subgroup B does not contain $N_G(H)$. Since B contains $H > C$, it is non-nilpotent and generous in G . So it follows from Theorem 3.13 that B is self-normalizing, and we obtain $H < B$.

We will denote by U and V respectively either $U_p(B)$ and $U_p(H)$, in case these two subgroups are non-trivial for a prime number p , or $U_0(B)$ and $U_{0,\bar{r}_0(B)}(H)$. In particular, we have $B = N_G(U)$. If H contains U , then $N_G(H)$ normalizes U , contradicting $B = N_G(U)$, hence H does not contain U . In particular we obtain $V < N_U(V)^\circ$ and $H < N_G(V)^\circ$. Now the maximality of H forces $N_G(V) = N_G(V)^\circ$. But, if V is non-trivial, then Fact 2.25 (5) shows that $N_G(V)^\circ$ does not contain $N_G(H)$, and since $N_G(H)$ normalizes V , this contradicts $N_G(V) = N_G(V)^\circ$. Hence V is trivial, and H' centralizes U by Fact 2.31 (7). Thus we have $N_G(H')^\circ > H$, and the maximality of H provides $N_G(H') = N_G(H')^\circ$. On the other hand, H' is non-trivial since $H > C$ is non-nilpotent. Then Fact 2.25 (5) implies $N_G(H) \not\leq N_G(H')^\circ$, contradicting that $N_G(H)$ normalizes H' and that $N_G(H') = N_G(H')^\circ$. This finishes our proof. \square

As a corollary, we obtain an improvement of Corollary 3.9.

Corollary 6.2. – *Let G be a minimal connected simple group with a non-trivial Weyl group, and let T be a non-trivial maximal p -torus of G for a prime p . Then $C_G(T)$ is a Carter subgroup of G .*

PROOF – Since $W(G)$ and T are non-trivial, $C_G(T)$ is not self-normalizing by Corollary 3.4. But Facts 2.25 (2) and 2.7 show that $C_G(T)$ contains a Carter subgroup C of G , so $C_G(T)$ is contained in $N_G(C)$ by Proposition 6.1. Since $C_G(T)$ is connected by Fact 2.24 (3), we obtain the result. \square

It follows from Proposition 6.1 that, if B is a Borel subgroup of a minimal connected simple group G and if B contains a Carter subgroup C of G , then $C = B \cap B^w \cap B^{w^2} \dots \cap B^{w^{n-1}}$ for each $w \in N_G(C) \setminus C$ such that $w^n \in C$ for $n \in \mathbb{N}$. The following theorem improves this evident conclusion and shows that in fact, $C = B \cap B^w$.

Theorem 6.3. – *Let B be a Borel subgroup of a minimal connected simple group G . If B contains a Carter subgroup C of G , then we have $C = B \cap B^w$ for each $w \in N_G(C) \setminus C$.*

PROOF – We assume toward a contradiction that $C \neq B \cap B^w$ for some $w \in N_G(C) \setminus C$. We may assume that, either B has a non-trivial q -unipotent subgroup for some prime q , or that is maximal for such a counterexample. In the latter case, we will set $r = \bar{r}_0(B)$.

Let now $H = B \cap B^w$. Containing a Carter subgroup of G strictly, the subgroup H cannot be abelian. Since $H \cap B^w$ is connected by Fact 2.25 (5), there is a Borel subgroup A containing $N_G(H')^\circ$. In particular, A contains H and C , and $A \cap A^w$ is connected by Fact 2.25 (5).

We show that $A \cap A^w = C$. If A (resp. B) has a non-trivial q -unipotent subgroup for a prime q , then Facts 2.34 (1) and 2.11 show that $A \cap A^w$ (resp. $B \cap B^w$) is abelian. Thus we obtain $A \cap A^w = C$ (resp. $B \cap B^w = C$). Then B has no non-trivial q -unipotent subgroup for any prime q , and we may assume that A has no non-trivial q -unipotent subgroup for any prime q . Now we have $r > 0$, and $r = \bar{r}_0(B)$ is maximal for such a counterexample B . We let $K = (A \cap B)^\circ$. In particular, K contains H , so A contains $C_G(K')^\circ \leq C_G(H')^\circ$. Thus, by Fact 2.36 (1), if B_3 and B_4 are distinct Borel subgroups containing K , then $(B_3 \cap B_4)^\circ = K$. Moreover, by Fact 2.11, we have $K' \leq F(A) \cap F(B)$, so A contains $C_G(F(A) \cap F(B))^\circ$, and Facts 2.34 (3) and (4) imply that $\bar{r}_0(A) > r$. By maximality of r , we obtain $A \cap A^w = C$.

Since $F(B)$ contains $H' \leq B'$ by Fact 2.11, the subgroup $Z(F(B))^\circ$ is contained in $N_G(H')^\circ \leq A$. In the same way, we have $Z(F(B^w))^\circ \leq A$, therefore $A \cap A^w$ contains $Z(F(B^w))^\circ$. Now C contains $Z(F(B^w))^\circ$, and since w normalizes C , we obtain $Z(F(B))^\circ \leq C$.

Now, let $U = U_{0,r}(C)$. By Fact 2.31 (1), $U_{0,r}(B)$ is nilpotent. Since Fact 2.32 (1) shows that $[B, U_{0,r}(B)]$ is a homogeneous $U_{0,r}$ -subgroup of B , we have either $U_{0,r}(B) \leq Z(B)$ or $U_{0,r}(Z(F(B))) \neq 1$. Since C contains $Z(F(B))$, this yields $U \neq 1$. But w normalizes U , so it normalizes $N_G(U)^\circ$ too. If $U = U_{0,r}(B)$, then we have $B = N_G(U)^\circ$. Otherwise we have $U < U_{0,r}(N_{U_{0,r}(B)}(U))$ by Fact 2.33 (1), so $C < N_G(U)^\circ$. In particular, this proves that $N_G(U)^\circ$ is not nilpotent.

On the other hand, $N_G(U)^\circ$ is connected, definable and contains a Carter subgroup C of G , and w normalizes $N_G(U)^\circ$, so Proposition 6.1 implies that w belongs to $N_G(U)^\circ$. But $N_G(U)^\circ$ contains C , and $N_G(U)^\circ$ is solvable since $U \neq 1$, hence C is self-normalizing in $N_G(U)^\circ$ by Fact 2.25 (5). This yields $w \in C$, contradicting our choice of w , so we obtain the result. \square

7. MAJOR BOREL SUBGROUPS

In this section, we will introduce and analyze the structure of a special class of Borel subgroups of minimal connected simple groups with a non-trivial Weyl group (*i.e.* of type (3) or (4)):

Definition 7.1. – *Let G be a group of finite Morley rank. A Borel subgroup B of G is said to be a major Borel subgroup if it satisfies the following conditions:*

- (1) *B is not nilpotent;*
- (2) *every Carter subgroup of B is contained in a Carter subgroup of G ;*
- (3) *for every non-nilpotent Borel subgroup A and Carter subgroup C of G such that $A \cap C$ contains a Carter subgroup of B , $\text{rk}(A \cap C) = \text{rk}(B \cap C)$.*

We start with a few remarks that may motivate this notion:

Remark 7.2. – If G is of type (4), then G has no nilpotent Borel subgroup by Lemma 3.11, so its major Borel subgroups are the ones containing a Carter subgroup of G . In this type of minimal connected simple groups, a Borel is major if and only if it is generous.

If G is of type (1), it is unclear that G has a non-nilpotent Borel subgroup as G could be a *bad group* (see [BN94, Chapter 13]).

If G is of type (3), then G has a non-nilpotent Borel subgroup because otherwise $W(G) = 1$ by Fact 2.13 (4); as a result, it has a major Borel subgroup. Moreover, such a subgroup contains no Carter subgroup of G by Theorem 4.1, and is not generous in G by Lemma 3.10.

The main result of this section is Theorem 7.7 that proves the existence of a factorization of major Borel subgroups in minimal simple groups with a non-trivial Weyl group in a way very reminiscent of the decomposition of connected solvable algebraic groups as semidirect product of their unipotent part by their maximal tori [Hum81, Theorem 19.3].

We start our analysis of major Borel subgroups of with those in minimal connected simple groups of type (3).

Lemma 7.3. – *Let G be a minimal connected simple group of type (3) and C be a Carter subgroup of G . Then there exists a Borel subgroup A of G such that $A \neq C$ and $A \cap C \neq 1$.*

PROOF – We assume toward a contradiction that $A \cap C$ is trivial for each Borel subgroup $A \neq C$. Every nilpotent Borel subgroup of G , being a Carter subgroup of G is conjugate to C by Fact 2.25 (4). Fact 2.13 (4) implies that G has non-nilpotent Borel subgroups since $W(G) \neq 1$. Thus, using the contradictory assumption, we conclude that G has a Borel subgroup that intersects every conjugate of C trivially. This conclusion allows us to build a Carter subgroup C_0 of G as in [FJ05], by considering the indecomposable subgroups of G not contained in $\cup_{g \in G} C^g$. But, since $A \cap C = 1$ for each Borel subgroup $A \neq C$, we obtain $C_0 \neq C^g$ for each $g \in G$. This contradicts Fact 2.25 (4). \square

Lemma 7.4. – *Let G be a minimal connected simple group of type (3). Then, for each Carter subgroup C of G and each Borel subgroup $B \neq C$, there is a Borel subgroup $A \neq C$ such that $A \cap C$ contains $B \cap C$ and is a Carter subgroup of A .*

Moreover, if $B \cap C$ has torsion or if $rk(B_0 \cap C) = rk(B \cap C)$ for each Borel subgroup $B_0 \neq C$ containing $B \cap C$, then $B \cap C$ is a Carter subgroup of B .

PROOF – First we note that C is a Borel subgroup by Theorem 4.1. Moreover, if $B \cap C$ is of finite index in $N_B(B \cap C)$, then Fact 2.25 (5) shows that $B \cap C$ is a Carter subgroup of B . So we may assume that $B \cap C$ is of infinite index in $N_B(B \cap C)$. By Lemma 7.3, we may assume that $B \cap C$ is non-trivial.

We assume toward a contradiction that the torsion part R of $B \cap C$ is non-trivial. If $U_p(C)$ is trivial for each prime p , then R is central in C by Fact 2.7. and $N_G(R)$ contains $N_G(B \cap C)$ and C . Since C is a Borel subgroup of G , this implies that $C = N_G(R)^\circ$ and that $B \cap C$ is of finite index in $N_B(B \cap C)$, contradicting that $B \cap C$ is of infinite index in $N_B(B \cap C)$. Therefore $U_p(C)$ is non-trivial for a prime p . As a result $U_p(C_C(R))$ is non-trivial by Fact 2.7, and C is the only Borel subgroup of G containing $N_G(R)^\circ$ by Fact 2.34 (1). Thus, once again we conclude that C contains $N_G(B \cap C)^\circ$ and thus $B \cap C$ is of finite index in $N_B(B \cap C)$, contradicting that $B \cap C$ is of infinite index in $N_B(B \cap C)$. Hence $B \cap C$ is torsion-free. In particular, $B \cap C$ is connected.

Now we may assume that, for each Borel subgroup $A \neq C$ containing $B \cap C$ we have $rk(A \cap C) = rk(B \cap C)$. We consider a Borel subgroup A containing $N_G(B \cap C)^\circ$. In particular, A contains $B \cap C$ by the previous paragraph. Then, since $B \cap C$ is of infinite index in $N_B(B \cap C)$, we have $rk(A \cap C) = rk(B \cap C)$. On the other hand, since $C > B \cap C$ is nilpotent, $B \cap C$ is of infinite index in $N_C(B \cap C) \leq A \cap C$. This contradiction finishes the proof. \square

Corollary 7.5. – *Let G be a minimal connected simple group of type (3). Then, for each Carter subgroup C of G and each Borel subgroup $B \neq C$, the subgroup $B \cap C$ is abelian and divisible.*

PROOF – By Lemma 7.4 and Fact 2.35 (2), the subgroup $B \cap C$ is connected and abelian. On the other hand, since $F(B) \cap C$ is torsion-free by Fact 2.34 (2), we have $U_p(B \cap C) = 1$ for each prime p . Thus, $B \cap C$ is divisible by Fact 2.6. \square

Proposition 7.6. – *Let G be a minimal connected simple group of type (3). Then the following two conditions are equivalent for any Borel subgroup B of G :*

- (1) *B is a major Borel subgroup;*
- (2) *there is a Carter subgroup $C \neq B$ of G such that, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $rk(A \cap C) = rk(B \cap C)$;*

In this case, $B \cap C$ is an abelian divisible Carter subgroup of B , and each Carter subgroup of B has the form $B \cap C^b$ for $b \in B$.

Moreover, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $A \cap C = B \cap C$.

PROOF – First we assume that B is a major Borel subgroup of B . Let D be a Carter subgroup of B . Then D is contained in a Carter subgroup C of G , and we have $C \neq B$ since B is non-nilpotent. Moreover, for each Borel subgroup $A \neq C$ containing $B \cap C$, either A is nilpotent or $rk(A \cap C) = rk(B \cap C)$. But Lemma 7.4 applied to A shows that $A \cap C$ is a Carter subgroup of A , so A is non-nilpotent, and we have $rk(A \cap C) = rk(B \cap C)$. Hence, since $A \cap C$ is connected by Corollary 7.5, we obtain $A \cap C = B \cap C$.

Now we assume that there is a Carter subgroup $C \neq B$ of G such that, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $rk(A \cap C) = rk(B \cap C)$. Then $B \cap C$ is a Carter subgroup of B by Lemma 7.4. In particular, B is non-nilpotent, and $B \cap C$ is abelian and divisible by Corollary 7.5. Moreover, Fact 2.25 (3) shows that any Carter subgroup of B has the form $B \cap C^b$ for $b \in B$. This implies the result. \square

Now, we can prove the main theorem of this section.

Theorem 7.7. – *Let B be a major Borel subgroup of a minimal connected simple group G with a non-trivial Weyl group, and let C be a Carter subgroup of G containing a Carter subgroup D of B . Then the following conditions are satisfied:*

$$D = B \cap C, \quad B = B' \rtimes D \quad \text{and} \quad Z(B) = F(B) \cap D.$$

Furthermore, B has the following properties:

- (1) *for each prime p , either $U_p(B')$ is the unique Sylow p -subgroup of B , or each Sylow p -subgroup of B is a p -torus contained in a conjugate of D ;*
- (2) *for each positive integer $r \leq \bar{r}_0(D)$, each Sylow $U_{0,r}$ -subgroup of B has the form $U_{0,r}(D^b)$ for $b \in B$.*

PROOF – We note that if G is of type (4), then B contains a Carter subgroup of G (Remark 7.2), and we have $D = C$ by Fact 2.25 (3), so $D = B \cap C$. If G is of type (3), then C is a Borel subgroup of G (Theorem 4.1) and B , despite being major, is relatively small. Nevertheless, as we will now show, it still controls the conjugacy of the Carter subgroups of G that it intersects non-trivially. By Proposition 7.6 there exists a Carter subgroup C_0 of G such that, for each Borel subgroup $A \neq C_0$ containing $B \cap C_0$, we have $rk(A \cap C_0) = rk(B \cap C_0)$, and that $D = B \cap C_0^b$ for some $b \in B$. We thus conclude that $rk(B_0 \cap C_0^b) = rk(B \cap C_0^b)$ for each Borel subgroup $B_0 \neq C_0^b$ containing $B \cap C_0^b$. Since C is a Borel subgroup of G that contains D , if $C \neq C_0^b$, then $rk(B_0 \cap C_0^b) = rk(C \cap C_0^b)$ for each Borel subgroup $B_0 \neq C_0^b$.

containing $C \cap C_0^b$. Thus by Lemma 7.4 $C \cap C_0^b$ is a proper Carter subgroup of C , a contradiction to the nilpotence of C . Hence we have $C = C_0^b$ and $D = B \cap C$. This argument also shows that, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $rk(A \cap C) = rk(B \cap C)$.

Now, by Fact 2.25 (6), we have $B = B'D$ and, by Fact 2.25 (5), we obtain $Z(B) \leq N_B(D) = D$, so $Z(B) \leq F(B) \cap D$. On the other hand, D is divisible and abelian by Theorem 5.2 and Corollary 7.5. We also remind that B/B' is divisible by Facts 2.11 and 2.12.

We verify assertion (1). Let p be a prime integer. We will show that, either $U_p(B')$ is the unique Sylow p -subgroup of B , or each Sylow p -subgroup of B is a p -torus contained in a conjugate of D . We may assume that $U_p(B')$ is not a Sylow p -subgroup of B . By Fact 2.29, there is no non-trivial p -torus in B' . It then follows from Facts 2.7 and 2.19 (1) that $U_p(B')$ is the Sylow p -subgroup of B' . Since $B = B'D$ and since D is abelian and divisible, the Sylow p -subgroup T of D is a non-trivial p -torus. Then, Facts 2.19 (2) and 2.14 (2) imply that there is a Sylow p -subgroup of B in $C_B(T)$. Moreover, by Corollary 6.2, $C = C_G(T)$. It follows from the preceding two conclusions that T is a Sylow p -subgroup of B , and (1) is then a consequence of Fact 2.18.

We note that, since D is abelian and divisible, assertion (1) implies that $B' \cap D$ is torsion-free.

Now we assume that $s = \bar{\tau}_0(D)$ is positive, and we consider a Sylow $U_{0,s}$ -subgroup S of G containing $U_{0,s}(D) = U_0(D)$. We suppose toward a contradiction that C does not contain S . We note that the hypothesis $s > 0$ implies that $U_0(D) \neq 1$. Let $R = U_{0,s}(S \cap C)$. If G is of type (4), we have $D = C$, so $R = U_0(D)$, and R is normal in $N_G(D)$, and D is not self-normalizing in $N_G(R)$ as we have $N_G(D)/D \simeq W(G) \neq 1$ by Proposition 3.2. On the other hand, Fact 2.33 (1) gives $R < U_{0,s}(N_S(R))$, and we obtain $D < N_G(R)^\circ$. Therefore Proposition 6.1 shows that $N_G(R)$ is a solvable connected subgroup of G . In particular D is self-normalizing in $N_G(R)$ (Fact 2.25 (5)), contradicting that D is not self-normalizing in $N_G(R)$. If G is of type (3), then Fact 2.31 (7) gives $D = U_0(D)C_D(U_{0,s}(C))$, so D normalizes R . Thus $N_G(R)^\circ$ contains D , and the maximality of the intersection $D = B \cap C$ implies either $N_C(R)^\circ = D$ and $R = U_0(D)$, or $N_G(R)^\circ \leq C$. But, as S is not contained in C , Fact 2.33 (1) implies $R < U_{0,s}(N_S(R))$, and we obtain $N_G(R)^\circ \not\leq C$, so we have $N_C(R)^\circ = D$ and $R = U_0(D)$. Consequently, we obtain $N_C(D)^\circ \leq N_C(U_0(D))^\circ = N_C(R)^\circ = D$, contradicting $D < N_C(D)^\circ$. Thus, in all the cases, $U_{0,s}(C)$ is the only $U_{0,s}$ -subgroup of G containing $U_0(D)$.

We assume toward a contradiction that there exists a positive integer $r \leq \bar{\tau}_0(D)$ such that $U_{0,r}(B')$ is non-trivial. Then, by Fact 2.31 (2), the subgroup $U_{0,r}(B')U_0(D)$ is nilpotent. On the other hand, by Facts 2.32 (2) and (3), there is a definable connected definably characteristic subgroup A of B' such that $B' = A \times U_{0,r}(B')$. But, since $U_{0,r}(B')$ is non-trivial, B/A is not abelian. Hence, since D is abelian and satisfies $B = B'D$, the group $U_{0,r}(B')$ is not contained in D . Now, in the case $r = \bar{\tau}_0(D)$, the group $U_{0,r}(B')U_0(D)$ is a nilpotent $U_{0,r}$ -subgroup of B containing the $U_{0,r}(D) = U_0(D)$ and not contained in C . Since this contradicts the previous paragraph, we obtain $r < \bar{\tau}_0(D)$, and by Fact 2.31 (7) $U_{0,r}(B')$ centralizes $U_0(D)$. In particular, this gives $U_{0,r}(B') \leq N_G(U_0(D))^\circ$. If G is of type (4), this yields $C < N_G(U_0(C))^\circ$, and Proposition 6.1 shows that $N_G(U_0(C))$ is a definable connected solvable subgroup of G . Since it contains $N_G(C)$, we have

a contradiction with Fact 2.25 (5) and Proposition 3.2. If G is of type (3), we have $D < N_C(D)^\circ \leq N_C(U_0(D))^\circ$, and the maximality of $D = B \cap C$ yields $N_G(U_0(D))^\circ \leq C$ and $U_{0,r}(B') \leq C$, contradicting $U_{0,r}(B') \not\leq D = B \cap C$. Consequently, in all the cases, $U_{0,r}(B')$ is trivial for each positive integer $r \leq \bar{\tau}_0(D)$.

We note that, since $B' \cap D$ is torsion-free, the last paragraph yields $B' \cap D = 1$ and $B = B' \rtimes D$. On the other hand, for each positive integer $r \leq \bar{\tau}_0(D)$, the group $[B, U_{0,r}(F(B))]$ is a homogeneous $U_{0,r}$ -group by Fact 2.32 (1), so $U_{0,r}(F(B))$ is central in B . Since the torsion part of $F(B) \cap D$ is central in B by Fact 2.21, we obtain $F(B) \cap D \leq Z(B)$ by (Fact 2.31 (7)). Thus $Z(B) = F(B) \cap D$, and the same holds for every Carter subgroup of B by Fact 2.25 (5).

Now we prove the assertion (2). Let $r \leq \bar{\tau}_0(D)$ be a positive integer, and let U be a Sylow $U_{0,r}$ -subgroup of B . Since $U_{0,r}(B')$ is trivial, by Fact 2.33 (3) there exists a Carter subgroup Q of B such that $U = U_{0,r}(Q)$. Hence assertion (2) follows from Fact 2.25 (3). \square

Corollary 7.8. – *Let B be a major Borel subgroup of a minimal connected simple group G with a non-trivial Weyl group, and let C be a Carter subgroup of G containing a Carter subgroup of B . If H is a subgroup of B containing a Carter subgroup D of B , then the following conditions are satisfied:*

$$H = H' \rtimes D \quad \text{and} \quad Z(H) = F(H) \cap D.$$

Furthermore, H has the following properties:

- (1) for each prime p , either $U_p(H')$ is the unique Sylow p -subgroup of H , or each Sylow p -subgroup of H is a p -torus contained in a conjugate of D ;
- (2) for each positive integer $r \leq \bar{\tau}_0(D)$, each Sylow $U_{0,r}$ -subgroup of H has the form $U_{0,r}(D^h)$ for $h \in H$.

PROOF – By Fact 2.25 (3) and Theorem 7.7, we may assume $D = B \cap C$. By Fact 2.25 (6), we have $H = H'D$, and Theorem 7.7 gives $H' \cap D \leq B' \cap D = 1$, so $H = H' \rtimes D$. In particular, we have $H' = B' \cap H$.

Now we prove the assertion (1). Let p be a prime, and let S be a Sylow p -subgroup of H . By Facts 2.19 (2) and 2.14 (2), we have $S = U_p(H) * T$ for a p -torus T . Then Theorem 7.7 (1) says that we have either $S = U_p(H) \leq U_p(B')$, or $S = T$. In the first case, we have $S \leq B' \cap H = H'$ and $S = U_p(H')$. In the second case, S is contained in a conjugate of D by Fact 2.25 (2) and (3). Now the conjugacy of Sylow p -subgroups in H yields (1).

We prove the second assertion. Let $r \leq \bar{\tau}_0(D)$ be a positive integer, and let S be a Sylow $U_{0,r}$ -subgroup of H . By Theorem 7.7 (2), we have $S \cap H' \leq S \cap B' = 1$, so Fact 2.33 (3) provides a Carter subgroup Q of H such that $U = U_{0,r}(Q)$. Hence the assertion (2) follows from Fact 2.25 (3).

From now on, we have just to prove the equality $Z(H) = F(H) \cap D$. By Fact 2.25 (5), we have $Z(H) \leq N_H(D) = D$, so $Z(H)$ is contained in $F(H) \cap D$. On the other hand, since $H = H' \rtimes D$, we have $F(H) = H' \times (F(H) \cap D)$, so the Sylow structure description of H obtained in the assertions (1) and (2), together with and Fact 2.31 (7), yields the conclusion. \square

8. JORDAN DECOMPOSITION

In this section and the next one, unless otherwise stated, G will denote a connected minimal simple group of finite Morley rank with a non-trivial Weyl group

(i.e. of type (3) or (4)). We denote by \mathcal{S} the union of its Carter subgroups and by \mathcal{U} its elements x satisfying $d(x) \cap \mathcal{S} = \{1\}$. The elements of \mathcal{S} are called *semisimple* and the ones of \mathcal{U} *unipotent*. This is the *Jordan decomposition* proposed in this article. It will have the same fundamental properties of the one in linear algebraic groups.

Remark 8.1. – If G is isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K , then its Carter subgroups are the maximal tori and its non-trivial unipotent subgroups have the form B' for B a Borel subgroup. Moreover, each element belongs to a maximal torus or a unipotent subgroup, hence our definitions of semisimple and unipotent elements coincide with the classical definitions in simple algebraic groups.

For each definable automorphism α of the pure group G , we have $\alpha(\mathcal{S}) = \mathcal{S}$ and $\alpha(\mathcal{U}) = \mathcal{U}$.

Since $W(G)$ is non-trivial, each Carter subgroup of G is generous (Theorem 4.1). Moreover, if G is of type (4), then Theorem 5.2 shows that each Carter subgroup of G is abelian and divisible.

Lemma 8.2. – *Let x be an element of a Carter subgroup C of G . Then one of the following three conditions is satisfied:*

- (A) *either $C_G(x)$ is connected;*
- (B) *or $C_G(x)$ is not connected, $C_G(x) \subseteq \mathcal{S}$ and one of the following holds:*
 - (1) *$|W(G)|$ is odd, G is of type (3), $C_G(x) \leq C$ and C is the only Borel subgroup of G that contains $C_G(x)$;*
 - (2) *$|W(G)| = 2$, $I(G) \neq \emptyset$, G is of odd type of Prüfer 2-rank 1, x is an involution and belongs to C , $C = C_G(x)^\circ$, $C_G(x) = C_G(x)^\circ \rtimes \langle i \rangle$ where $i \in I(G)$ and inverts $C_G(x)^\circ$.*

PROOF – We may assume that $C_G(x)$ is not connected. By Corollary 3.9, we have $C = C_G(T)$ for a maximal decent torus T of G . First we assume that $|W(G)|$ is even. We may assume that G is not isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K . Then Fact 2.16, the classification of simple groups of even type, and Fact 3.6 imply that G is of odd type and of Prüfer 2-rank one. It follows from Fact 3.6 that $|W(G)| = 2$, that involutions of G are conjugate, and that G has an abelian Borel subgroup D such that $N_G(D) = D \rtimes \langle i \rangle$ for an involution i inverting D . By the conjugacy of C and D (Fact 2.25 (4)), we obtain $C_G(x) = N_G(C) = C \rtimes \langle j \rangle$ for an involution j inverting C . In particular, x is an involution, and the elements of jC are involutions, which are semisimple by conjugacy. Hence we may assume that $|W(G)|$ is odd.

In addition to $|W(G)|$'s being odd, we also assume that G is of type (3). In this paragraph and the next, we analyze the consequences of these hypotheses. In particular, C is a nilpotent Borel subgroup by Theorem 4.1. For each prime p and each p -element $a \in N_G(C) \setminus C$, the prime p divides $|W(G)|$, and by Fact 3.7 there is no non-trivial p -torus in T . Then Fact 3.8 (2) implies $a \notin C_G(x)$, and we conclude $C_G(x) \cap N_G(C) \leq C$. Thus, if C is the only Borel subgroup containing $C_G(x)^\circ$, we obtain $C_G(x) \leq N_G(C_G(x)^\circ) \leq N_G(C)$ and $C_G(x) \leq C$, so we may assume that there is a Borel subgroup $B \neq C$ containing $C_G(x)^\circ$. We will show that this leads to a contradiction. If $U_p(C)$ is non-trivial for a prime p , then $U_p(Z(C)) \leq C_G(x)$ is non-trivial too by Fact 2.7, and Fact 2.34 (1) say that C is the only Borel subgroup containing $C_G(x)^\circ$, contradicting the previous sentence. Hence $U_p(C)$ is trivial

for each prime p , and $C_C(x)$ contains the torsion of C by Fact 2.7, so $C_C(x)$ is connected and B contains $C_C(x)$.

Now, for each Borel subgroup $B_0 \neq C$ containing $B \cap C \geq C_C(x)$, since $B_0 \cap C$ is abelian (Corollary 7.5), we have $B_0 \cap C = C_C(x) = B \cap C \geq C_G(x)^\circ \geq C_C(x)$, so $B \cap C = C_G(x)^\circ = C_C(x)$ is a maximal intersection between C and another Borel subgroup. It follows from these two conclusions and Lemma 7.4 that $C_G(x) \leq N_G(C_G(x)^\circ) \leq N_G(C)$. But, we have already proven that $C_G(x) \cap N_G(C) \leq C$. This contradicts the assumption that $C_G(x)$ is not connected.

Finally, we will show that G is not of type (4) when $|W(G)|$ is odd. If, toward a contradiction, G is of type (4), then C is abelian, $C_G(x)$ contains C , and we have $C_G(x)^\circ = C$ by Proposition 6.1. Then there is a prime p dividing $|C_G(x)/C|$. In particular, p divides $|W(G)|$ by Proposition 3.2. We consider a p -element a in $C_G(x) \setminus C$. Since $C = C_G(T)$, we obtain $a \in N_G(T)$ and $x \in C_{C_G(T)}(a) \setminus \{1\}$. Then Facts 3.7 and 3.8 (2) yield a contradiction. \square

Corollary 8.3. – *If G is of type (3), let C be a Carter subgroup of G , and let B be a Borel subgroup subject to one of the following conditions:*

- (\dagger) *B contains $N_G(U)^\circ$, where U is a definable connected subgroup of C ;*
- ($\dagger\dagger$) *B contains $C_G(x)^\circ$ where $x \in C$.*

Then either $B = C$ or B is a major Borel subgroup. In the latter case, $B \cap C$ is a Carter subgroup of B contained in H , where H is either $N_G(U)^\circ$ as in (\dagger) or $C_G(x)^\circ$ as in ($\dagger\dagger$).

In the case where $H = C_G(x)^\circ$ with $x \in C$, we have $x \in B$.

PROOF – In the case where $H = C_G(x)^\circ$ with $x \in C$, $x \in B$ by Lemma 8.2. Thus in both cases, since $B \cap C$ is abelian and divisible by Corollary 7.5, we have $B \cap C \leq H$. Let $A \neq C$ be a Borel subgroup containing $B \cap C$. Similarly $A \cap C \leq H$, and $rk(A \cap C) = rk(H \cap C) = rk(B \cap C)$. Proposition 7.6 yields the result. \square

Lemma 8.4. – *Let B be a major Borel subgroup of G , and let C be a Carter subgroup of G such that $D = B \cap C$ is a Carter subgroup of B . Let H be a subgroup of B containing D . Then we have $H \cap \mathcal{U} = H'$ and, for each element x of H , there exists $(x_u, x_s) \in (\mathcal{U} \cap d(x)) \times (\mathcal{S} \cap d(x))$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$.*

Furthermore, if A is any subset of H formed by some semisimple elements and generating a nilpotent subgroup, then A is conjugate in H with a subset of D . In particular, we have $H \cap \mathcal{S} = \cup_{h \in H} D^h$.

PROOF – By the Sylow structure description of H obtained in Corollary 7.8, we have $H' \cap \mathcal{S} = \{1\}$, so H' is contained in $H \cap \mathcal{U}$.

We show that, for each element x of H , there exist $h \in H$ and $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$. By Fact 2.28, the generalized centralizer $E_H(x)$ of x in H is definable and connected, x belongs to its Fitting subgroup $F(E_H(x))$, and, by Facts 2.26 (1) and 2.27, $E_H(x)$ contains a Carter subgroup Q of H . Moreover, there exists $h \in H$ such that $Q = D^h$ (Fact 2.25 (3)), and Corollary 7.8 yields $F(E_H(x)) = E_H(x)' \times Z(E_H(x))$ and $Z(E_H(x)) = F(E_H(x)) \cap D^h$. It follows from Fact 2.6 that $d(x) = d(x)^\circ \times U$ with U a finite cyclic subgroup, and $d(x)^\circ$ divisible. Also, by Fact 2.31 (6), if T denotes the maximal decent torus of $d(x)$, then $d(x)^\circ$ is the product of T by its Sylow $U_{0,r}$ -subgroups for all the positive integers r . Let π be the set of primes p such that

$E_H(x)'$ has a non-trivial p -element, and let π' be its complementary in the set of primes. Let S_1 be the set of π -elements of $d(x)$ and let S_2 be the set of π' -elements of $d(x)$. Then Corollary 7.8 (1) gives $S_1 \leq E_H(x)'$ and $S_2 \leq Z(E_H(x))$. Moreover, we have $T \leq d(S_1)d(S_2)$. Also, Corollary 7.8 (2) shows that, for each positive integer r , we have either $U_{0,r}(d(x)) \leq E_H(x)'$ or $U_{0,r}(d(x)) \leq Z(E_H(x))$. This implies $d(x) = (d(x) \cap E_H(x)') \times (d(x) \cap Z(E_H(x)))$. Since $E_H(x)'$ is contained in H' , and since $Z(E_H(x))$ is contained in D^h , we obtain $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$.

Note that, since \mathcal{U} contains H' , we have $x_u \in \mathcal{U} \cap d(x)$. On the other hand, since \mathcal{S} contains $D^h \leq C^h$, we have $x_s \in \mathcal{S} \cap d(x)$, and if x is semisimple, then we obtain $x_u = 1$, and $x = x_s$ belongs to $D^h \subseteq \cup_{k \in H} D^k$. This implies the equality $H \cap \mathcal{S} = \cup_{k \in H} D^k$.

Now let $x \in H \setminus H'$. By the previous paragraph, there exists $h \in H$ and $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ such that $x = x_u x_s$. In particular, x_s is a non-trivial semisimple element of $d(x)$, so x is not unipotent, and we obtain $H \cap \mathcal{U} = H'$.

Let A be a subset of H formed by some semisimple elements and generating a nilpotent subgroup. Then, by Fact 2.28, the generalized centralizer $E_H(A)$ of A in H is definable and connected, $F(E_H(A))$ contains A and, by Facts 2.26 (1) and 2.27, there is a Carter subgroup P of H in $E_H(A)$. Moreover, since there exists $h \in H$ such that $P = D^h$ (Fact 2.25 (3)), Corollary 7.8 yields $F(E_H(A)) = E_H(A)' \times Z(E_H(A))$ and $Z(E_H(A)) = F(E_H(A)) \cap P$. But, by previous paragraphs, the semisimple elements of $E_H(A)$ are contained in $\cup_{k \in E_H(A)} P^k$. Thus, the ones in $F(E_H(A))$ are central in $E_H(A)$. Hence A is contained in a central subgroup of $E_H(A)$, and we obtain $A \subseteq D^h$, as desired. \square

Theorem 8.5. – (Jordan decomposition)

- (1) For each $x \in G$, there exists a unique $(x_s, x_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x_s x_u = x_u x_s$.
- (2) For each $x \in G$, we have $d(x) = d(x_s) \times d(x_u)$.
- (3) For each $(x, y) \in G \times G$ such that $xy = yx$, we have $(xy)_u = x_u y_u$ and $(xy)_s = x_s y_s$.

PROOF – We first prove (1) and (2). Let $x \in G$. We show that there exists $(x_s, x_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x_s x_u = x_u x_s$, and such that $d(x) = d(x_s) \times d(x_u)$. We may assume that x is neither semisimple, nor unipotent. In particular, there exists $y \in d(x) \setminus \{1\}$ such that y belongs to a Carter subgroup C_0 of G . Since $x \in C_G(y)$ is not semisimple, Lemma 8.2 shows that $C_G(y)$ is connected. Then, if G is of type (4), we have $C_G(y) \geq C_0$ as C_0 is abelian, and Lemma 8.4 proves the existence of (x_s, x_u) . If G is of type (3), then as $C_G(y)$ contains an element that is not semisimple, by Corollary 8.3 there exists a major Borel subgroup B_y containing $C_G(y)$ and such that $B_y \cap C_0$ is a Carter subgroup of B_y . Hence, the existence of (x_s, x_u) follows from Lemma 8.4.

Now we show that, for each $(x'_s, x'_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x'_s x'_u = x'_u x'_s$, we have $(x'_s, x'_u) = (x_s, x_u)$. First we assume $x_s = 1$. Then we have $x = x_u$ and we may assume $x'_s \neq 1$. If $C_G(x'_s)$ is not connected, then Lemma 8.2 gives $C_G(x'_s) \subseteq \mathcal{S}$. But we have $x = x_u = x'_s x'_u = x'_u x'_s$, hence x and x'_u are two unipotent elements in $C_G(x'_s)$. Therefore we have $x = x'_u = 1$, so $x'_s = 1$, contradicting $x'_s \neq 1$. Consequently $C_G(x'_s)$ is connected and not contained in \mathcal{S} . Since $x = x_u$ and x'_u are two unipotent elements of $C_G(x'_s)$, they belong to $C_G(x'_s)'$ by Corollary 8.3 and

Lemma 8.4. On the other hand, x'_s is a non-trivial semisimple element, so Lemma 8.4 gives $x'_s \notin C_G(x'_s)'$, contradicting $x = x'_u x'_s$. Hence we may assume $x_s \neq 1$.

Let C be a Carter subgroup of G containing x_s . If we have $x_u = x'_u = 1$, then we obtain $x = x_s = x'_s$ and $(x'_s, x'_u) = (x_s, x_u)$. Hence we may assume that x_u or x'_u is non-trivial. In particular, $C_G(x_s)$ contains a nonsemisimple element, and by Lemma 8.2, $C_G(x_s)$ is connected. Thus, if G is of type (3), then Corollary 8.3 provides a major Borel subgroup $B_x \neq C$ containing $C_G(x_s)$ and such that $B_x \cap C$ is a Carter subgroup of B_x contained in $C_G(x_s)$. Now we may apply Corollary 7.8 and Lemma 8.4 in $C_G(x_s)$. On the other hand, if G is of type (4), then we have $D = C \leq C_G(x_s)$, and we may apply Corollary 7.8 and Lemma 8.4 in $C_G(x_s)$ too. Now, in all the cases, Lemma 8.4, provides $h \in C_G(x_s)$ such that $x'_s \in D^h$. Moreover, Lemma 8.4 gives $x_u \in C_G(x_s)'$ and $x'_u \in C_G(x_s)'$, and since x_s is central in $C_G(x_s)$, we obtain $x = x_u x_s \in F(C_G(x_s))$. But, since we have $x'_u \in C_G(x_s)'$, we obtain $x'_s = x(x'_u)^{-1} \in F(C_G(x_s))$ too. Hence Corollary 7.8 gives $x'_s \in F(C_G(x_s)) \cap D^h = Z(C_G(x_s))$. From now on, we have $(x_u, x_s) \in C_G(x_s)' \times Z(C_G(x_s))$ and $(x'_u, x'_s) \in C_G(x_s)' \times Z(C_G(x_s))$, so as Corollary 7.8 shows that $F(C_G(x_s))$ is the internal direct product of $C_G(x_s)'$ by $Z(C_G(x_s))$, we obtain $(x_u, x_s) = (x'_u, x'_s)$. This finishes the proof of (1) and (2).

In order to prove (3), it suffices to prove that the product of two commuting semisimple (resp. unipotent) elements is semisimple (resp. unipotent). In this vein, suppose that x and y are two non-trivial semisimple elements that commute. We may assume $C_G(x) \not\subseteq \mathcal{S}$. In particular, Lemma 8.2 implies that $C_G(x)$ is connected. Then, by using Corollary 8.3 when G is of type (3), we may apply Lemma 8.4 in $C_G(x)$, and we find a Carter subgroup of G that contains both x and y . Now suppose that x and y are two non-trivial unipotent elements that commute. We may assume $(xy)_s \neq 1$. Then by Lemma 8.2 $C_G((xy)_s)$ is connected and not contained in \mathcal{S} . Indeed, as $xy = (xy)_s(xy)_u$ such that $(xy)_s$ and $(xy)_u$ commute, either $(xy)_u \neq 1$ and $C_G((xy)_s) \not\subseteq \mathcal{S}$, or $xy = (xy)_s$. In the latter case, we still conclude $C_G((xy)_s) \not\subseteq \mathcal{S}$ because x and y commute with xy , therefore with $(xy)_s$ which is equal to xy . As a result, by using Corollary 8.3 when G is of type (3), we may apply Lemma 8.4 in $C_G((xy)_s)$. It follows that x and y belong to $C_G((xy)_s)' \subseteq \mathcal{U}$, and the proof of (3) is finished. \square

9. CONSEQUENCES ON THE STRUCTURE OF BOREL SUBGROUPS

In this section, we will continue the analysis of minimal simple connected groups of type (3) and (4) along the lines determined by the Jordan decomposition introduced in the last section. We will gradually develop an analysis of various families of subgroups of a simple group of type (3) or (4). This will proceed from Sylow subgroups to arbitrary definable connected solvable non-nilpotent subgroups, including a visit to the nilpotent world (Subsection 9.2) and the introduction of a new notion of torus (Subsection 9.3). The analysis will culminate in Theorem 9.15 that takes to a higher level of generality the conclusions of Theorem 7.7 and Corollary 7.8. Theorem 9.15 has an important precursor that shows the relevance of our notion of torus, namely Theorem 9.12.

Our standing assumption on the notation will remain invariant: G is a minimal connected simple group with non-trivial Weyl group, equivalently of types (3) or (4). For each subgroup H of G , we denote by H_u the set $H \cap \mathcal{U}$ of its unipotent elements, and by H_s the set $H \cap \mathcal{S}$ of its semisimple elements.

9.1. Sylow subgroups. It is well-known that in an algebraic group, the characteristic of the underlying field plays a decisive role on the nature of torsion elements, and this phenomenon is observed through the use of the Jordan decomposition in that torsion elements are either semisimple or unipotent. In Proposition 9.2, we will obtain a similar result for minimal connected simple groups with a non-trivial Weyl group by proving that the Sylow p -subgroups of G are not of mixed type, in the sense that each Sylow p -subgroup is contained either in \mathcal{U} or in \mathcal{S} . However, in a minimal connected simple group, it is not clear whether the elements of a p -unipotent group are unipotent, a well-known property of connected simple algebraic groups over algebraically closed fields (cf. Proposition 9.2 (2) (a)).

Another well-known property in the algebraic category is that in minimal connected simple algebraic groups over algebraically closed fields, equivalently in $\mathrm{PSL}_2(K)$ with K algebraically closed the semisimple/unipotent dichotomy becomes global since every non-trivial element is either semisimple or unipotent. In Proposition 9.4, we will exhibit an analogous behaviour in the context of minimal connected simple groups, by proving a result similar to Proposition 9.2 for the Sylow $U_{0,r}$ -subgroups of G .

The following conclusion from [BD09], in the spirit of Fact 2.19 (2), will be handy:

Fact 9.1. – [BD09, Corollary 4.7] *Let G be a minimal connected simple group and p a prime different from 2. Then the maximal p -subgroups of G are connected.*

Proposition 9.2. – *Let p be a prime number, and let S be a Sylow p -subgroup of G . Then one of the following three conditions is satisfied:*

- (1) $S \subseteq \mathcal{U}$ and S is p -unipotent;
- (2) $S \subseteq \mathcal{S}$, S is contained in a Carter subgroup C of G , and it is connected; furthermore, we have two possibilities:
 - (a) G is of type (3) and $S \cap B$ is a p -torus of Prüfer p -rank at most 1 for each Borel subgroup $B \neq C$;
 - (b) G is of type (4) and S is a p -torus of Prüfer p -rank at most 2;
- (3) $S \subseteq \mathcal{S}$, $p = 2$, S° is a 2-torus of Prüfer 2-rank one, and $S = S^\circ \rtimes \langle i \rangle$ for an involution i inverting S° .

PROOF – We may assume that G is not isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K . If $p = 2$, by Fact 2.16, the classification of simple groups of even type and Fact 3.6, the group S° is a non-trivial 2-torus, and one of the following two conditions is satisfied:

- (†) $|W(G)| = 2$, S° is a 2-torus of Prüfer 2-rank one, the involutions of G are conjugate, and G has an abelian Borel subgroup C_0 such that $N_G(C_0) = C_0 \rtimes \langle i \rangle$ for an involution i inverting C_0 ;
- (††) $|W(G)| = 3$ and S° is a 2-torus of Prüfer 2-rank two.

The group S° is a maximal 2-torus of G , and even a maximal connected 2-subgroup of G by Fact 2.15. By Corollary 6.2, $C_G(S^\circ)$ is a Carter subgroup of G . In particular, S° is the only Sylow 2-subgroup of $C_G(S^\circ)$ by Fact 2.19 (1), so $N_G(S^\circ) = N_G(C_G(S^\circ))$. Thus, in case (†), Fact 2.25 (4) yields an involution j inverting $C_G(S^\circ)$ and such that $N_G(S^\circ) = C_G(S^\circ) \rtimes \langle j \rangle$. Then, by conjugacy of the Sylow 2-subgroups in $N_G(S^\circ)$ (Fact 2.15), we may decompose S in the form $S = S^\circ \rtimes \langle k \rangle$ for an involution k inverting S° . Moreover, since S° is a 2-torus, the elements

of the coset kS° are some involutions, which are semisimple by conjugacy of the involutions in G . Hence S satisfies the assertion (3).

In case $(\dagger\dagger)$, by Corollary 3.4 $N_G(S^\circ)/C_G(S^\circ) \simeq W(G)$ has order 3, so $S \leq C_G(S^\circ)$. In particular, $S = S^\circ$ is connected and it is contained in the Carter subgroup $C_G(S^\circ)$ of G . On the other hand, the Carter subgroups of G are not Borel subgroups by Fact 3.6, consequently G is of type (4) by Theorem 4.1, and S satisfies the assertion (2) (b) of our result. Hence we may assume $p \neq 2$.

We first show that if S is a p -unipotent subgroup then S satisfies (1) or (2) (a). We may assume that S contains a non-trivial semisimple element x . By Fact 2.34 (1), there is a unique Borel subgroup B of G containing $Z(S)^\circ$. In particular, B contains S and $C_G(x)^\circ$ and, by Fact 2.18, there is no non-trivial p -torus in B . Thus, x centralizes no non-trivial p -torus. If G is of type (4), then the Carter subgroups are abelian and divisible by Theorem 5.2, and x belongs to a non-trivial p -torus. This contradicts that there is no non-trivial p -torus in $B \geq C_G(x)^\circ$. Hence G is of type (3). Then, by Corollary 8.3, if C denotes a Carter subgroup containing x , we have either $B = C$ or B is a major Borel subgroup containing x , and $B \cap C$ is a Carter subgroup of B . In the latter case, $B \cap C$ is abelian and divisible by Corollary 7.5. Hence, $x \in B \cap C$ belongs to a p -torus. This is contradictory since there is no non-trivial p -torus in B . Hence we find $B = C$, and C contains no non-trivial p -torus. Since, for each Borel subgroup $B_0 \neq C$, the group $B_0 \cap C$ is abelian and divisible by Corollary 7.5, this implies that $B_0 \cap B = B_0 \cap C$ has no non-trivial p -element, so $S \cap B_0 = 1$. Thus S satisfies (2) (a), as desired.

From now on, we may assume that S is not a p -unipotent subgroup. By Fact 9.1, S is connected. By Fact 2.14 (2), the maximal p -torus T of S is non-trivial, and $C_G(T)$ contains S . By Fact 3.7, p does not divide $|W(G)|$. By Corollary 6.2, $C_G(T)$ is a Carter subgroup of G that we will more simply denote as C_T .

If G is of type (4), Corollary 7.8 (1) shows that S is a p -torus. Also, this p -torus has Prüfer p -rank at most 2 by Theorem 5.2, hence S satisfies (2) (b). Therefore we may assume that G is of type (3). In particular, C_T is not only a Carter subgroup but also a Borel subgroup of G containing S . Let $B \neq C_T$ be another Borel subgroup. We show that $S \cap B$ is a p -torus of Prüfer p -rank at most 1. By Lemma 7.4 and Proposition 7.6, we may assume that B is a major Borel subgroup, and that $B \cap C_T$ is a Carter subgroup of B . Let A be a B -minimal subgroup in B' . By Theorem 7.7, we have $A \cap C_T \leq B' \cap C_T = 1$, so $B \cap C_T$ does not centralize A . Consequently, Fact 2.10 provides a definable algebraically closed field K such that $(B \cap C_T)/C_{B \cap C_T}(A)$ is definably isomorphic to a subgroup of the multiplicative group K^* . By Corollary 7.5, $S \cap B$ is a p -torus. If $\text{pr}_p(S \cap B) \geq 2$, then there is a non-trivial p -torus S_0 in $C_{B \cap C_T}(A)$. By Fact 2.7, S_0 centralizes C_T . It follows that $C_G(S_0)^\circ$ is a proper definable subgroup of G containing C_T and A . This contradicts that C_T is a Borel subgroup of G . Hence, $\text{pr}_p(S \cap B) = 1$ and S satisfies (2) (a). \square

Corollary 9.3. – *Let S be a Sylow p -subgroup of a solvable connected definable subgroup H of G . If H is non-nilpotent, then one of the following two conditions is satisfied:*

- (1) $S \subseteq \mathcal{U}$ and S is p -unipotent;
- (2) $S \subseteq \mathcal{S}$ and S is a p -torus of Prüfer p -rank at most 2.

PROOF – Since S is connected by Fact 2.19 (2), the result follows from Fact 2.14 (2) and from Proposition 9.2. \square

As was mentioned at the beginning of this subsection, by a different argument, we obtain a similar result for Sylow $U_{0,r}$ -subgroups, where r is a positive integer.

Proposition 9.4. – *For each positive integer r and each Sylow $U_{0,r}$ -subgroup S of G , one of the following two conditions is satisfied:*

- (1) $S \subseteq \mathcal{U}$ and S is a homogeneous $U_{0,r}$ -subgroup;
- (2) $S \subseteq \mathcal{S}$ and S is contained in a unique Carter subgroup of G .

PROOF – First, we assume $S \subseteq \mathcal{U}$, and prove that S is a homogeneous $U_{0,r}$ -subgroup. By Fact 2.25 (2), for each prime p , there is no non-trivial p -torus in S , and Fact 2.7 implies that S is torsion-free. We consider the subgroup S^* generated by the indecomposable subgroups A of S satisfying $rk(A/J(A)) \neq r$. In other words, S^* is generated by the subgroups of the form $U_{0,s}(S)$ for $s \neq r$. We will show that $S^* = \{1\}$. In this vein, we assume that S^* is non-trivial. By Fact 2.32 (1), the groups of the form $[N_G(S)^\circ, U_{0,s}(S)]$, where s is a positive integer, are some homogeneous $U_{0,s}$ -subgroups. Since S is a $U_{0,r}$ -subgroup, they are $U_{0,r}$ -subgroup too. Hence $N_G(S)^\circ$ centralizes S^* .

On the other hand, $N_G(S)^\circ$ is a subgroup of $N_G(S^*)^\circ$ that contains a Carter subgroup D of $N_G(S^*)^\circ$ by Fact 2.33 (4). We show that $S^* = U_{0,r}(D)^*$, where $U_{0,r}(D)^*$ is the subgroup generated by the indecomposable subgroups A of $U_{0,r}(D)$ satisfying $rk(A/J(A)) \neq r$. Since S is the unique Sylow $U_{0,r}$ -subgroup of $N_G(S)^\circ$ by Fact 2.33 (2), we have $U_{0,r}(D) \leq S$ and $U_{0,r}(D)^* \leq S^*$. In order to prove that $U_{0,r}(D)^*$ contains S^* , we have just to verify that $U_{0,r}(D)$ contains S^* . But D centralizes $S/[D, S]$, so $DS/[D, S]$ is a nilpotent group and Fact 2.25 (6) gives $DS = [D, S]D$. Hence we have $S = [D, S](S \cap D)$ and since $[D, S]$ is a homogeneous $U_{0,r}$ -subgroup by Fact 2.32 (1), we obtain $S = [D, S]U_{0,r}(S \cap D)$ by Fact 2.31 (5). The homogeneity of $[D, S]$ implies $S \cap D = ([D, S] \cap D)U_{0,r}(S \cap D) = U_{0,r}(S \cap D)$, and since $[N_G(S)^\circ, S^*] = 1$, S^* is contained in D and thus in $S \cap D = U_{0,r}(S \cap D) \leq U_{0,r}(D)$. This is what was desired and proves that $S^* = U_{0,r}(D)^*$.

The previous paragraph implies that $N_G(D)^\circ$ normalizes S^* , so D is a Carter subgroup of G and $S^* \leq D$ is contained in \mathcal{S} . Consequently we have $S^* \subseteq S \cap \mathcal{S} \subseteq \mathcal{U} \cap \mathcal{S} = \{1\}$, and S is homogeneous.

From now on, we may assume that there is a Carter subgroup C of G with $S \cap C \neq 1$, and we have to prove that S is contained in a conjugate of C . We assume toward a contradiction that S is contained in no Carter subgroup of G . We may assume that C is chosen such that $rk(U_{0,r}(S \cap C))$ is maximal. We will now verify that $U_{0,r}(S \cap C) = 1$ and that as a result $[S, S \cap C] = 1$ (Fact 2.32 (1)). If $U_{0,r}(S \cap C)$ is non-trivial, we consider a Borel subgroup B containing $N_G(U_{0,r}(S \cap C))^\circ$. Then Fact 2.31 (4) gives $U_{0,r}(S \cap C) < U_{0,r}(S \cap B)$ and, by maximality of $rk(U_{0,r}(S \cap C))$, the subgroup $U_{0,r}(S \cap B)$ is contained in no conjugate of C . In particular, if G is of type (3), then we have $B \neq C$ and Corollary 8.3 says that B is a major Borel subgroup such that $B \cap C$ is a Carter subgroup of B . If G is of type (4), then B contains C and B is a major Borel subgroup too. Hence, in all the cases, Theorem 7.7 (2) gives $r > \bar{\tau}_0(B \cap C)$, contradicting that $U_{0,r}(S \cap C)$ is non-trivial. Thus $U_{0,r}(S \cap C)$ is trivial, and by Fact 2.32 (4) S centralizes $S \cap C$.

Let $x \in (S \cap C) \setminus \{1\}$, and let B be a Borel subgroup containing $C_G(x)^\circ$. In particular, B contains S , and we have $B \neq C$. Then, if G is of type (3), Corollary 8.3 says that B is a major Borel subgroup and that $B \cap C$ is a Carter subgroup of B . On the other hand, if G is of type (4), we have $C \leq B$ and B is a major Borel subgroup too. Thus, in both cases, since S is contained in no Carter subgroup of

G , Theorem 7.7 (2) gives $r > \bar{r}_0(B \cap C)$ and $B = B' \rtimes (B \cap C)$. This implies $S \leq B'$ and $S \cap C = 1$, contradicting $S \cap C \neq 1$. Hence S is contained in a conjugate of C , and we may assume $S \leq C$.

We will prove that no other Carter subgroup of G contains S . Let $w \in N_G(C) \setminus C$. Then, since C normalizes $S = U_{0,r}(C)$, we have $w \in N_G(S)$, and Theorem 6.3 provides $C = N_G(S)^\circ \cap (N_G(S)^\circ)^w$. But $w \in N_G(S)$ normalizes $N_G(S)^\circ$, hence we obtain $C = N_G(S)^\circ$. Moreover, this equality is true for each Carter subgroup of G containing S , so C is the unique Carter subgroup containing S . \square

The previous result has the following consequence on the conjugacy of the Sylow $U_{0,r}$ -subgroups.

Corollary 9.5. – *Let r be a positive integer, and let S be a Sylow $U_{0,r}$ -subgroup of G . Then S is conjugate with any Sylow $U_{0,r}$ -subgroup R of G satisfying $S \cap R \neq 1$.*

PROOF – We assume toward a contradiction that R is a counterexample with $rk(S \cap R)$ maximal. In particular, by nilpotence of S and R , we have $S \cap R < N_S(S \cap R)$ and $S \cap R < N_R(S \cap R)$. Moreover, by Proposition 9.4 and by Fact 2.25 (4), the $U_{0,r}$ -subgroups S and R are contained in \mathcal{U} and they are homogeneous. Thus $S \cap R$ is a $U_{0,r}$ -subgroup.

Let $H = N_G(S \cap R)^\circ$ and let S_1 (resp. R_1) be a Sylow $U_{0,r}$ -subgroup of H containing $S \cap H$ (resp. $R \cap H$). By Fact 2.33 (2), there exists $h \in H$ such that $R_1^h = S_1$. Let S_2 be a Sylow $U_{0,r}$ -subgroup of G containing S_1 . Since $S \cap H > S \cap R$ is contained in $S \cap S_2$, there exists $g \in G$ such that $S_2^g = S$ by maximality of $rk(S \cap R)$. Then we obtain

$$(S \cap R)^{hg} < (R \cap H)^{hg} \leq R_1^{hg} = S_1^g \leq S_2^g = S.$$

But this forces

$$rk(S \cap R) < rk((R \cap H)^{hg}) \leq rk(R^{hg} \cap S).$$

Thus, R^{hg} and S are conjugate by maximality of $rk(S \cap R)$, a contradiction to our choice of R . \square

9.2. Structure of nilpotent subgroups. The following result is similar to a classical result for algebraic groups [Hum81, Proposition 19.2].

Proposition 9.6. – *For each nilpotent definable subgroup H of G , the sets H_u and H_s are two definable subgroups satisfying $H = H_u \times H_s$.*

Moreover, either H_s is contained in a Carter subgroup of G , or $H = H_s$ is a finite 2-subgroup contained in no Borel subgroup.

PROOF – First we assume that $Z(H)$ is not contained in \mathcal{U} , and we consider a non-trivial semisimple element x in $Z(H)$. Then $C_G(x)$ contains H . If $C_G(x)$ is not connected, then Lemma 8.2 gives $H = H_s$, and says that either H_s is contained in a Carter subgroup of G , or G is of odd type and of Prüfer 2-rank one, x is an involution, $C_G(x)^\circ$ is a Carter subgroup of G , and $C_G(x) = C_G(x)^\circ \rtimes \langle i \rangle$ for an involution i inverting $C_G(x)^\circ$. We may assume that we are in the second case, and that H is not contained in $C_G(x)^\circ$. Then we have $H = (H \cap C_G(x)^\circ) \rtimes \langle j \rangle$ for an involution j inverting $H \cap C_G(x)^\circ$. It follows from this that H is a finite 2-group. Indeed, if $z \in Z(H) \cap C_G(x)^\circ$, then $z = z^j = z^{-1}$, and $z^2 = 1$. Thus $Z(H)$ is an elementary abelian 2-group. But G is of odd type. Thus $Z(H)$ is finite. It follows from Fact 2.9 (2) that H is finite. Moreover, H has only 2-torsion elements since,

H being nilpotent, any non-trivial Sylow p -subgroup intersects $Z(H)$ non-trivially. Since $x \in C_G(x)^\circ$ by Lemma 8.2, x and j are two distinct involutions of H , and they commute. Therefore, if H is contained in a Borel subgroup B of G , then the Sylow 2-subgroups of B are 2-tori of Prüfer 2-rank one since they are connected by Fact 2.19 (2), non-trivial, and since G has Prüfer 2-rank one. This contradicts that x and j belongs to B . Hence H is contained in no Borel subgroup of G , as desired. Thus we may suppose that $C_G(x)$ is connected.

Let C be a Carter subgroup of G containing x , and let B be a Borel subgroup containing $C_G(x)$. Then either G is of type (3), and Corollary 8.3 says that B is a major Borel subgroup such that $B \cap C$ is a Carter subgroup of B , or G is of type (4), and B is a major Borel subgroup containing C . Consequently, Lemma 8.4 says that H_s is conjugate in $C_G(x)$ with a subset of C , and we may assume $H_s \subseteq C$. This implies that C contains $d(H_s)$, so H_s is a definable subgroup of H . On the other hand, $H_u \subseteq C_G(x)' \subseteq \mathcal{U}$ by Lemma 8.4 gives, so $C_G(x)'$ contains $d(H_u)$ and H_u is a definable subgroup of H . Now the equality $H = H_u \times H_s$ follows from the Jordan decomposition of each element of H (Theorem 8.5 (1) and (2)).

It remains the case when $Z(H)$ is contained in \mathcal{U} . We will prove that $H \subseteq \mathcal{U}$. By contradiction, we suppose that H is not contained in \mathcal{U} . Then we find $x \in H_s \setminus \{1\}$, and we may assume that x is chosen such that $C_H(x)$ is maximal for such an element x . By the previous paragraphs, $C_H(x)_u$ and $C_H(x)_s$ are two definable subgroups satisfying $C_H(x) = C_H(x)_u \times C_H(x)_s$. In particular, since $Z(H)$ is contained in \mathcal{U} , we have $C_H(x) < H$, and we obtain $C_H(x) < N_H(C_H(x))$. Since $C_H(x)_s$ is definably characteristic in $C_H(x)$, $N_H(C_H(x))$ normalizes $C_H(x)_s$, and there exists a non-trivial element z in $Z(N_H(C_H(x))) \cap C_H(x)_s$. Hence z is a non-trivial semisimple element of H such that $C_H(x) < N_H(C_H(x)) \leq C_H(z)$, which contradicts the maximality of $C_H(x)$. The proof is finished. \square

9.3. Tori. We start this subsection by introducing a notion of torus generalizing the algebraic ones. We will call a *torus*, any definable connected subgroup T of G satisfying $T = T_s$.

Proposition 9.7. – *The maximal tori of G are Carter subgroups. In particular, they are conjugate and, if G is of type (4), they are abelian.*

PROOF – Since G is of type (3) or (4) by our standing assumption, it has a major Borel subgroup B_0 . Hence B'_0 is a non-trivial subgroup of G contained in \mathcal{U} by Lemma 8.4, and thus G is not a torus. Consequently, the tori of G are solvable.

We consider a Carter subgroup C of G . By the previous paragraph, if G is of type (3), then C is a maximal torus. If G is of type (4), then there is a maximal torus T containing C . The elements of T' are unipotent by Lemma 8.4, and so T is abelian. Consequently we obtain $T = C$, and each Carter subgroup of G is a maximal torus.

Now, since the Carter subgroups of G are conjugate by Fact 2.25 (4) and they are abelian when G is of type (4), it remains to prove that each torus of G is contained in a Carter subgroup of G . Let T be a torus of G . If T is nilpotent, then it is contained in a Carter subgroup of G by Proposition 9.6, so we may assume that T is not nilpotent. Then T' is a non-trivial nilpotent torus by Fact 2.11, and T' is contained in a Carter subgroup C of G by Proposition 9.6. Let $H = N_G(T')^\circ$. Then H is a solvable non-nilpotent connected subgroup of G containing T . If G is of type (3), then Corollary 8.3 and Lemma 8.4 give $T' \leq H' \subseteq \mathcal{U}$, contradicting that

T' is a non-trivial torus. So G is of type (4), and H contains C since C is abelian. Therefore we obtain $T' \leq H' \subseteq \mathcal{U}$ again, contradicting that T' is a non-trivial torus. Consequently, the maximal tori of G are Carter subgroups. \square

Lemma 9.8. – *Let H be a definable connected solvable subgroup of G . Then either H is a torus, or $F(H)_s$ is a central subgroup of H .*

PROOF – We may assume that H is not a torus. By Proposition 9.6, $F(H)_s$ is a definable subgroup of a Carter subgroup C of G . We notice that we have $H \not\leq C$ since H is not a torus. Let x be a non-trivial p -element of $F(H)_s$ for a prime p , and let S be a Sylow p -subgroup of H containing x . Then S is a p -torus by Corollary 7.5 (in case G is of type (3)), Proposition 9.2 and Fact 2.19 (2), and x is central in H by Fact 2.21. Thus, to finish, it will suffice to prove that $F(H)_s^\circ$ is central in H . We may assume $F(H)_s^\circ \neq 1$.

Let B be a Borel subgroup of G containing $N_G(F(H)_s)^\circ$. Since H normalizes $F(H)_s$, it will suffice to prove that $F(H)_s \leq Z(N_G(F(H)_s)^\circ)$. If G is of type (4), then C is abelian, so $C \leq N_G(F(H)_s)^\circ$ and B is a major Borel subgroup. It follows from Corollary 7.8 that $F(H)_s \leq F(N_G(F(H)_s)^\circ) \cap C = Z(N_G(F(H)_s)^\circ)$.

We finish the proof handling the case when G is of type (3). Since H is not a torus and $H \leq N_G(F(H)_s)^\circ \leq B$, necessarily $B \neq C$. Hence, by Corollary 8.3 B is a major Borel subgroup of G , and $B \cap C$ is a Carter subgroup of B contained in $N_G(F(H)_s)^\circ$. Corollary 7.8 allows to finish as above. \square

Corollary 9.9. – *Let H be a definable connected solvable subgroup of G . If $F(H)_s$ is non-trivial, then either H is a torus, or H is contained in a major Borel subgroup.*

PROOF – We may assume that H is not a torus. Let $x \in F(H)_s \setminus \{1\}$. Therefore $C_G(x)^\circ$ contains H by Lemma 9.8. Now let B be a Borel subgroup containing $C_G(x)^\circ$. If G is of type (3), then by Corollary 8.3, B is a major Borel subgroup; if G is of type (4), then any Carter subgroup of G containing x is in $C_G(x)^\circ \leq B$. The result follows. \square

Lemma 9.10. – *Let H be a solvable connected definable subgroup of G . If R is a subgroup of H formed by semisimple elements, then there is a Carter subgroup D of H such that R is contained in D_s .*

PROOF – We may assume that R is non-trivial, and that R is maximal among the subgroups of H formed by some semisimple elements of H . Moreover, we may assume that H is non-nilpotent by Proposition 9.6. So H is not a torus by Proposition 9.7. Then, since $F(H)_s$ is a subgroup of H by Proposition 9.6, and that it is central in H by Lemma 9.8, Theorem 8.5 (3) implies $F(H)_s \leq R$ by maximality of R , and in fact obtain $F(H)_s \leq Z(R)$. Now R is nilpotent since R' is contained in $F(H)_s$ by Fact 2.11.

We let $E = E_H(R)$. Since by Fact 2.28 E is a connected definable subgroup of H and that $F(E)$ contains R , we have $R = F(E)_s$ by Proposition 9.6 and by maximality of R . Let D be a Carter subgroup of E (Fact 2.25 (1)). If E is nilpotent, we have $E = D = F(E)$ and $R = D_s$. If not, then E is not a torus by Proposition 9.7, and $R \leq Z(E)$ by Lemma 9.8. Fact 2.25 (5) then implies that $R \leq D$. Again we conclude $R = D_s$ by Proposition 9.6 and by maximality of R . Since by Facts 2.26 (2), 2.27 and 2.25 (3), D is a Carter subgroup of H , we obtain the result. \square

The conjugacy of maximal tori in H now follows from Fact 2.25 (3):

Corollary 9.11. – *In each proper definable connected subgroup H of G , the maximal tori of H are conjugate.*

Theorem 9.12. – *In each connected solvable definable subgroup H of G , the set H_u is a connected definable subgroup such that $H = H_u \rtimes T$ for any maximal torus T of H .*

In particular, unless G is of type (3) and H is a torus, we have $H' \subseteq H_u$.

PROOF – First we notice that, unless G is of type (3) and H is a torus, any torus of H is abelian by Corollary 7.5 and Proposition 9.7. So it will suffice to prove that H_u is a connected definable subgroup such that $H = H_u \rtimes T$ for any maximal torus T of H . In particular, we may assume that H is not a torus.

We claim that H_u contains H' . Since $F(H)$ contains H' by Fact 2.11, we may assume that H is contained in a major Borel subgroup by Corollary 9.9, and we obtain $H' \subseteq H_u$ by Corollary 7.8 and Proposition 9.6.

On the other hand, if T is any maximal torus of H , then Proposition 9.6 and Lemma 9.10 provide a Carter subgroup D of H such that $T = D_s$ and $D = D_u \rtimes T$. Moreover, Fact 2.25 (6) gives $H = H'D = (H'D_u)T$. Thus, since D_u is definable and connected by Proposition 9.6, it remains to prove that $H'D_u = H_u$.

We claim that the subgroup $H'D_u$ contains only unipotent elements. Suppose towards a contradiction that there exists $x \in (\mathcal{S} \cap H'D_u) \setminus \{1\}$. By Lemma 9.10 and by conjugacy of Carter subgroups (Fact 2.25 (3)), we may assume $x \in T$. Then we have $x = hd$ for $h \in H' \subseteq \mathcal{U}$ and $d \in D_u \subseteq \mathcal{U}$. This implies $h = xd^{-1}$. Since $xd^{-1} = d^{-1}x$ with $x \in \mathcal{S}$ and $d^{-1} \in \mathcal{U}$, we obtain a contradiction to the Jordan decomposition of $h \in \mathcal{U}$ (Theorem 8.5 (1)).

The preceding paragraphs show that $(H'D_u) \subseteq H_u$. We will show now that these two sets are in fact equal. Indeed, for each $x \in H_u$ then, by Facts 2.28, 2.27, and 2.26 (2) the set $E_H(x)$ is a definable connected subgroup containing a Carter subgroup of H , and such that x belongs to $F(E_H(x))$. By Fact 2.25 (3), we may assume $D \leq E_H(x)$. Since $H = H'D$, we have $x = hd$ for $d \in D$ and $h \in H' \cap E_H(x) \subseteq F(E_H(x))_u$. In particular, this implies $d \in F(E_H(x))$. But, by Proposition 9.6, the set $F(E_H(x))_u$ is a subgroup of $F(E_H(x))$. Hence, since x belongs to $F(E_H(x))_u$ as well, we conclude $d \in F(E_H(x))_u$, and $d \in D \cap \mathcal{U} = D_u \leq H'D_u$. This yields $x = hd \in H'D_u$ and $H_u = H'D_u$. \square

9.4. Structure of solvable subgroups. It is unclear if, in Theorem 9.12, the subgroup H_u is nilpotent. We will clarify this in this section of which Theorem 9.15 is the main conclusion. It incorporates all the developments up to this point in this article, in particular the Jordan decomposition.

Lemma 9.13. – *Let B be a Borel subgroup of G . If $B \subseteq \mathcal{U}$, then B is torsion-free.*

PROOF – By Fact 2.25 (2), each decent torus of B is trivial. Consequently, using Facts 2.14 (2) and 2.19 (2), we may assume that $U_p(B)$ is non-trivial for a prime p . We let $U = U_p(B)$. If a B -minimal section \bar{A} of U is not centralized by B , then $B/C_B(\bar{A})$ is definably isomorphic to a definable subgroup of K^* for a definable algebraically closed field K of characteristic p by Fact 2.10, and Fact 2.22 shows that $B/C_B(\bar{A})$ is a decent torus. Then there is a non-trivial decent torus in B by Fact 2.24 (1), contradicting that each decent torus of B is trivial. Consequently each B -minimal section of U is centralized by B . This implies that, if C denotes a Carter subgroup of B , then C contains U , so $U = U_p(C)$.

Since $B \subseteq \mathcal{U}$, C is not a Carter subgroup of G by the definition of a semisimple element. Hence B does not contain $N_G(C)^\circ$. On the other hand, we have proven that $B = N_G(U)^\circ \geq N_G(C)^\circ$. This contradiction finishes the proof. \square

Lemma 9.14. – *Let r be a positive integer, and let S be a Sylow $U_{0,r}$ -subgroup of G . If $S \subseteq \mathcal{U}$, then $B = N_G(S)^\circ$ is a Borel subgroup of G , and S is contained in B' .*

PROOF – First we note that S is a homogeneous $U_{0,r}$ -group by Proposition 9.4. Also, if S is contained in B' for a Borel subgroup B of G , the nilpotence of B' (Fact 2.11) as well as the unipotent structure of nilpotent groups of finite Morley rank (Facts 2.31 (6), (7) and 2.32 (2)) imply that $S = U_{0,r}(B')$ is normal in B and that $B = N_G(S)^\circ$. Then we may assume that, for each Borel subgroup B of G , we have $S \not\subseteq B'$. We will assume towards a contradiction that r is a minimal counterexample to the statement of the lemma. Thus for each positive integer $s < r$ and for each $U_{0,s}$ -Sylow subgroup R of G , the condition $R \subseteq \mathcal{U}$ implies the existence of a Borel subgroup A of G satisfying $R \leq A'$.

As a first step, we show that, for each Borel subgroup B of G such that $S \cap B$ is non-trivial, no Sylow $U_{0,r}$ -subgroup of B is contained in B' . Indeed, by Fact 2.33 (2) and Corollary 9.5, we may assume that $S \cap B$ is a Sylow $U_{0,r}$ -subgroup of B , and that $S \cap B$ is contained in B' . Then, the nilpotence of B' (Fact 2.11) and the unipotent structure of nilpotent groups of finite Morley rank (Facts 2.31 (6), (7) and 2.32 (2)) imply that $S \cap B = U_{0,r}(B')$ is normal in B and that $B = N_G(S \cap B)^\circ$. By nilpotence of S , we obtain $S \leq B'$, contradicting our choice of S . Hence, no Sylow $U_{0,r}$ -subgroup of B is contained in B' .

The second main step of the proof will consist in showing that $B \cap \mathcal{S} = \{1\}$ for each Borel subgroup B of G such that $S \cap B$ is non-trivial. We assume toward a contradiction that B is a Borel subgroup of G such that $B \cap \mathcal{S}$ and $S \cap B$ are non-trivial. Since S is homogeneous, we may assume that $S \cap B$ is a Sylow $U_{0,r}$ -subgroup of B by Corollary 9.5. By the previous paragraph, $S \cap B$ is not contained in B' . By Fact 2.33 (4) there exists a Carter subgroup D of B in $N_B(S \cap B)^\circ$, and D_s is non-trivial by Lemma 9.10 and Fact 2.25 (3). Since D centralizes $(S \cap B)/[D, S \cap B]$, Fact 2.25 (6) gives $D(S \cap B) = [D, S \cap B]D$ and $S \cap B = [D, S \cap B](S \cap D)$. But $S \cap B$ is not contained in B' , hence $S \cap D$ is non-trivial. Let $x \in D_s \setminus \{1\}$. Then, by Proposition 9.6, we have $S \cap D \leq D_u \leq C_G(x)^\circ$. Moreover, $C_G(x)^\circ$ is contained in a major Borel subgroup A . Indeed, if G is of type (3), then we have $C_G(x)^\circ \not\subseteq \mathcal{S}$ since $C_G(x)^\circ \geq S \cap D \neq 1$, and Corollary 8.3 justifies the existence of A ; on the other hand, if G is of type (4), then since Carter subgroups are abelian, A exists. Since $S \cap A \geq S \cap D$ is non-trivial, by Corollary 9.5 there exists $g \in G$ such that $S^g \cap A$ is a Sylow $U_{0,r}$ -subgroup of A . Then, since $S \subseteq \mathcal{U}$, Lemma 8.4 yields $S^g \cap A \leq A'$, and contradicts the first step. Thus we have $B \cap \mathcal{S} = \{1\}$ for each Borel subgroup B of G such that $S \cap B$ is non-trivial. In particular, B is torsion-free by Lemma 9.13.

In the final step, we consider the smallest positive integer s such that there exists a Borel subgroup B with $S \cap B \neq 1$ and $U_{0,s}(B) \neq 1$. Then we fix such a Borel subgroup B whose Sylow $U_{0,s}$ -subgroups have maximal Morley rank. By Corollary 9.5, we may choose B such that $S \cap B$ is a Sylow $U_{0,r}$ -subgroup of B . In particular, by the first step, $S \cap B$ is not contained in B' . Also, by Facts 2.33 (2) and (3) there is a Carter subgroup D of B such that $U_{0,r}(D) = S \cap D$ and $S \cap B = (S \cap B')(S \cap D)$, so $S \cap D$ is non-trivial. Since s is minimal and B is torsion-free by the second step, $U_{0,s}(B')D$ is nilpotent by Fact 2.31 (2) and $U_{0,s}(D)$ is a Sylow $U_{0,s}$ -subgroup of B

by Fact 2.33 (3). We consider a Borel subgroup A of G containing $N_G(U_{0,s}(D))^\circ$. Then A contains D , so $S \cap A$ is non-trivial, and it follows from the second step that A is torsion-free. Moreover, the choice of s implies that $U_{0,t}(A)$ is trivial for each positive integer $t < s$. Since $U_{0,s}(D)$ is a Sylow $U_{0,s}$ -subgroup of B contained in A , the choice of B implies that $U_{0,s}(D)$ is a Sylow $U_{0,s}$ -subgroup of A too. Consequently, there is a Carter subgroup C of A in $N_A(U_{0,s}(D))$ by Fact 2.33 (4) and C contains $U_{0,s}(D)$ by Fact 2.31 (2). Now we have $U_{0,s}(C) = U_{0,s}(D)$, and $N_G(C)^\circ$ is contained in $N_G(U_{0,s}(C))^\circ = N_G(U_{0,s}(D))^\circ \leq A$, so C is a Carter subgroup of G . This contradicts the second step which implies $A \cap \mathcal{S} = \{1\}$, and completes the proof. \square

Theorem 9.15. – *Any Carter subgroup D of a non-nilpotent Borel subgroup B of G is abelian, divisible, and satisfies*

$$B = B' \rtimes D \quad \text{and} \quad Z(B) = F(B) \cap D.$$

Furthermore, B has the following properties:

- (1) for each prime p , either $U_p(B')$ is the unique Sylow p -subgroup of B , or each Sylow p -subgroup of B is a p -torus contained in a conjugate of D ;
- (2) there is at most one positive integer $r \leq \bar{\tau}_0(D)$ such that there is a Sylow $U_{0,r}$ -subgroup S of B not of the form $U_{0,r}(D^b)$ for $b \in B$. In this case, S is a maximal abelian $U_{0,r}$ -subgroup and is not a Sylow $U_{0,r}$ -subgroup of G .

PROOF – We note that, by Theorem 7.7, we may assume that B is not a major Borel subgroup of G . In particular, D is not a Carter subgroup of G . Moreover, Theorem 9.12 shows that B' is contained in \mathcal{U} , and more strongly, Corollary 9.9 gives $F(B) \subseteq \mathcal{U}$.

First we show that D is divisible. If D is not divisible, then by Fact 2.7 $U_p(D) \neq 1$ for a prime p . By Lemma 9.13, we have $B \not\subseteq \mathcal{U}$. Fact 2.25 (3) and Lemma 9.10 imply $D_s \neq 1$. By Proposition 9.6, D_s is a connected definable subgroup of a Carter subgroup C of G , and D_s centralizes $U_p(D) \subseteq D \cap \mathcal{U}$, so D_s centralizes $U_p(D)$ by Proposition 9.2. Moreover, by Corollary 8.3 if G is of type (3), and by the commutativity of Carter subgroups if G is of type (4), we have $N_G(D_s)^\circ \not\subseteq B$ since B is not a major Borel subgroup. But Fact 2.34 (1) says that B is the only Borel subgroup containing $N_G(D_s)^\circ \geq U_p(D) \neq 1$, hence we have a contradiction, and D is divisible.

Secondly, D is abelian. Indeed, $D < N_G(D)^\circ$, and the conclusion follows from Fact 2.35 (2).

Thirdly, we show that $B = B' \rtimes D$. By Fact 2.25 (6), we have $B = B'D$, and DB''/B'' is a Carter subgroup of B/B'' . Then, since D is abelian, Fact 2.25 (7) yields $B/B'' = B'/B \rtimes DB''/B''$, therefore $D \cap B'$ is contained in B'' . By Facts 2.32 (2) and (3), we have

$$B' = A \times U_{0,1}(B') \times \cdots \times U_{0,\bar{\tau}_0(B')}(B'),$$

where A is definable, connected, definably characteristic and of bounded exponent, and where $U_{0,s}(B')$ is a homogeneous $U_{0,s}$ -subgroup for each $s \in \{1, 2, \dots, \bar{\tau}_0(B')\}$. If $D \cap A$ is non-trivial, there is a prime p such that $U_p(B')$ is non-trivial and, since D is abelian and divisible, D contains a non-trivial p -torus T . Then $U_p(B')T$ is a locally finite p -subgroup of G contradicting Corollary 9.3. Hence $D \cap A$ is trivial, and we may assume that $D \cap U_{0,r}(B')$ is non-trivial for a positive integer r . We notice that, since B' is contained in \mathcal{U} , each Sylow $U_{0,r}$ -subgroup of B is contained

in \mathcal{U} by Fact 2.33 (2) and Proposition 9.4. On the other hand, since $D \cap B'$ is contained in B'' , the structure of B' implies that $D \cap U_{0,r}(B')'$ is non-trivial. So B is the unique Borel subgroup containing $U_{0,r}(B')$ by Fact 2.35 (2), and $U_{0,r}(B')$ is a Sylow $U_{0,r}$ -subgroup of G by Lemma 9.14 and Proposition 9.4 (1). Since D is not a Carter subgroup of G , we have $N_G(D)^\circ \not\leq B$, and $N_G(U_{0,r}(D))^\circ$ is contained in a Borel subgroup $A \neq B$. In particular, D is contained in A and is not a Carter subgroup of A . Let $S = N_{U_{0,r}(B')}(U_{0,r}(D))^\circ$. Then $S \leq A \cap B$ is abelian by Fact 2.35 (2), and since S contains $C_{U_{0,r}(B')}(U_{0,r}(D))^\circ$, it is a maximal abelian subgroup of $U_{0,r}(B')$. On the other hand, $D \cap U_{0,r}(B')'$ is non-trivial, so $U_{0,r}(B')$ is not abelian and we have $S < N_{U_{0,r}(B')}(S)^\circ$. By maximality of S in $U_{0,r}(B')$, the group $N_{U_{0,r}(B')}(S)^\circ$ is not abelian. This implies that B is the only Borel subgroup containing $N_G(S)^\circ$ Fact 2.35 (2). Now, if S_A is a Sylow $U_{0,r}$ -subgroup of A containing S , then S_A is a homogeneous $U_{0,r}$ -subgroup by Proposition 9.4 (1), and $N_{S_A}(S)^\circ$ is a $U_{0,r}$ -subgroup. But $N_{S_A}(S)^\circ \leq N_G(S)^\circ$ is contained in B , hence it is contained in $U_{0,r}(B)$. Since $U_{0,r}(B')$ is a Sylow $U_{0,r}$ -subgroup of G and that it is normal in B , we have $U_{0,r}(B) = U_{0,r}(B')$ by Fact 2.33 (2) and $N_{S_A}(S)^\circ$ is contained in $U_{0,r}(B')$. Thus, since S is a maximal abelian subgroup of $U_{0,r}(B')$, and since $N_{S_A}(S)^\circ \leq A \cap B$ is abelian Fact 2.35 (2), we obtain $N_{S_A}(S)^\circ = S$. Therefore the nilpotence of S_A yields $S_A = S$ and S is a Sylow $U_{0,r}$ -subgroup of A . Consequently, $N_G(S)^\circ$ contains a Carter subgroup of A by Fact 2.33 (4) and all the Carter subgroups of $N_G(S)^\circ$ are Carter subgroups of A (Fact 2.25 (3)). In particular, D is a Carter subgroup of A , contradicting that D is not a Carter subgroup of A . This proves $B = B' \rtimes D$.

Now we show that $Z(B) = F(B) \cap D$. Since D is abelian and divisible, for each prime p , each p -element x of $F(B) \cap D$ lies in a p -torus, and is semisimple by Fact 2.25 (2). Since $F(B)$ is contained in \mathcal{U} , this implies that $F(B) \cap D$ is torsion-free. On the other hand, for each positive integer r , if U is a non-trivial $U_{0,r}$ -subgroup in $F(B) \cap D$, then U is contained in the Sylow $U_{0,r}$ -subgroup S of $F(B) \subseteq \mathcal{U}$. Since $S \geq U$ is not contained in B' , there is a Borel subgroup $B_0 \neq B$ containing S by Lemma 9.14. In particular, S is abelian by Fact 2.35 (2), and S is central in $F(B)$ by Fact 2.31 (7). Consequently, since D is abelian, U centralizes $F(B)$ and D . Hence U is central in B . Therefore Fact 2.31 (7) provides $F(B) \cap D \leq Z(B)$, and the equality $Z(B) = F(B) \cap D$ follows from Fact 2.25 (5).

We verify assertion (1). Let p be a prime integer. If there is a p -element in $B \setminus B'$, then there is a non-trivial p -element in $D \simeq B/B'$. Since D is abelian and divisible, the maximal p -torus T of D contains all the p -elements of D . But Fact 2.25 (2) and (3) imply that T is a maximal p -torus of B , and Corollary 9.3 says that T is a Sylow p -subgroup of B . Hence the conjugacy of Sylow p -subgroups in B (Fact 2.18) allows to conclude (1) in this case. Thus we may assume that all the p -elements of B are contained in $B' \subseteq \mathcal{U}$, and Corollary 9.3 finishes the proof of (1).

Finally, we prove assertion (2). We may assume $\bar{\tau}_0(D) > 0$. Let A be a Borel subgroup containing $N_G(U_0(D))^\circ \geq N_G(D)^\circ > D$. In particular, we have $A \neq B$. By Fact 2.35 (1), there is a positive integer r such that $((A \cap B)^\circ)'$ is a homogeneous $U_{0,r}$ -subgroup. Let $s \leq \bar{\tau}_0(D)$, and let S be a Sylow $U_{0,s}$ -subgroup of B . By Fact 2.33 (3), there is a Carter subgroup Q of B such that $S = U_{0,s}(B')U_{0,s}(Q)$. By Fact 2.25 (3), $Q = D^b$ for $b \in B$. On the other hand, by Fact 2.31 (2), the subgroup $SU_0(D^b)$ is nilpotent. If $s < \bar{\tau}_0(D)$, then $U_0(D^b)$ centralizes S Fact 2.31 (6), and

$S \leq B \cap A^b$ is abelian by Fact 2.35 (2). If $s = \bar{\tau}_0(D)$ and $U_0(D) \subseteq S$, then S is contained in a Carter subgroup of G by Proposition 9.4 (2), and, since B is not major, S is abelian by Fact 2.35 (2). If $s = \bar{\tau}_0(D)$ and $U_0(D) \not\subseteq S$, then we have $S \subseteq \mathcal{U}$ by Proposition 9.4 (1). In this case, S is contained in B'_S for a Borel B_S (Lemma 9.14). Since $s = \bar{\tau}_0(D) > 0$ and $D \cap B' = 1$, we have $B_S \neq B$. Again Fact 2.35 (2) implies that S is abelian. Thus, in all the cases, S is abelian and centralizes $U_0(D^b)$. Then S is contained in $(A^b \cap B)^\circ$. Let now, $H = (A^b \cap B)^\circ$. Since D^b is a Carter subgroup of H , we have $S = U_{0,s}(H')U_{0,s}(D^b)$ by Fact 2.33 (3). Hence, since $H' = (((A \cap B)^\circ)')^b$ is a homogeneous $U_{0,r}$ -subgroup, $s = r$ whenever $H' \neq 1$. In particular, this proves the uniqueness statement in assertion (2). We may assume $H' \neq 1$.

In order to complete the proof, it remains to prove that S is a maximal abelian $U_{0,r}$ -subgroup and is not a Sylow $U_{0,r}$ -subgroup of G . Before going any further, we verify that H is a maximal intersection of Borels in G with respect to containment. We will use condition (ii) of Fact 2.36 (1) to verify this. Since S is an abelian Sylow $U_{0,r}$ -subgroup of B , all the Sylow $U_{0,r}$ -subgroups of B are abelian by Fact 2.33 (2), and the Sylow $U_{0,r}$ -subgroup of $F(B)$ is central in $F(B)$ by Fact 2.31 (7). Thus, since $F(B)$ contains B' by Fact 2.11, the $U_{0,r}$ -group H' centralizes B' . On the other hand, since $D^b \leq H$, D^b normalizes H' , and so $B = B' \rtimes D^b$ normalizes H' . This implies that $B = N_G(H')^\circ$. In particular, $B \geq C_G(H')^\circ$. The maximality follows.

An immediate consequence of the last paragraph is that S is a maximal abelian $U_{0,r}$ -subgroup of G . Indeed, if S_A is a maximal abelian $U_{0,r}$ -subgroup of G containing S , then $S_A \leq C_G(S)^\circ \leq C_G(H')^\circ \leq B$. Thus, $S = S_A$ by maximality of S in B .

It remains to prove that S is not a Sylow $U_{0,r}$ -subgroup of G . Before proceeding towards this conclusion, we verify that B' does not contain S . If B' contains S , then S is normal in B and $B = N_G(S)^\circ$. By Fact 2.33 (1), S is a Sylow $U_{0,r}$ -subgroup of G . Then $N_{A^b}(S)^\circ \leq B$ contains a Carter subgroup C_{A^b} of A^b by Fact 2.33 (4), and C_{A^b} is a Carter subgroup of H . Thus C_{A^b} and D^b are conjugate in H (Fact 2.25 (3)), and D^b is a Carter subgroup of A^b , contradicting that D is not a Carter subgroup of A . Hence B' does not contain S .

Finally, assume towards a contradiction that S is a Sylow $U_{0,r}$ -subgroup of G . Since $H' \leq B' \subseteq \mathcal{U}$, $S \subseteq \mathcal{U}$ by Proposition 9.4 (1). Let $B_S = N_G(S)^\circ$. By Lemma 9.14, B_S is a Borel subgroup of G satisfying $S \leq B'_S$. It then follows using the conclusion of the preceding paragraph that $B \neq B_S$. Since $H' \leq S \leq H$, $H \leq N_G(S)^\circ = B_S$. Hence, $B \cap B_S$ is also a maximal intersection. Since $B \geq N_G(H')^\circ$, Fact 2.36 (2) implies that $\bar{\tau}_0(B) > \bar{\tau}_0(B_S)$. Since S is a Sylow $U_{0,r}$ -subgroup of G , S is abelian and $S \triangleleft B_S$, we conclude that $S = U_{0,r}(F(B_S))$. Fact 2.36 (3) yields a contradiction. \square

10. REDUCTS

This section is a pleasant detour motivated by questions of more model-theoretic nature. We analyze the robustness of various notions introduced in this article with respect to reducts.

From the model-theoretic viewpoint, a group is in general not only the *pure group structure*, that is a group regarded as an \mathcal{L} -structure where \mathcal{L} is the language

of groups, but a structure definable in richer structures with interesting model-theoretic properties. Such a “definable” group inherits additional structural properties from the ambient structure. To what extent the mere language of groups is powerful enough to recover the additional structure is a recurrent question relevant for groups of finite Morley rank as well. This section aims at providing answers to this general question for various concepts fundamental for this article.

Lemma 10.1. – *Let G be a minimal connected simple group of finite Morley rank, and let $W(G)$ be its Weyl group. Then the pure group G is a minimal connected simple group of finite Morley rank too, and its Weyl group is $W(G)$.*

PROOF – Since G is a connected simple group of finite Morley rank, then the pure group G is a connected simple group of finite Morley rank too. Moreover, if B denotes a maximal proper connected definable subgroup of the pure group G , then B is a proper definable subgroup of G , so B is solvable-by-finite. Therefore, since it is connected relatively to the pure group G , it is solvable, and the pure group G is a minimal connected simple group of finite Morley rank.

Moreover, it follows from Corollary 3.4 that the Weyl group of the pure group G is $W(G)$. \square

Proposition 10.2. – *Let G be a minimal connected simple group of finite Morley rank. Then the Carter subgroup of G are the ones of the pure group G .*

In particular, the semisimple elements of G are preserved by all the automorphisms of the pure group G .

PROOF – Let C be a Carter subgroup of G . Then C is a maximal nilpotent subgroup of G by Corollary 3.3. But, in the pure group G , the definable closure of C is a nilpotent subgroup of G containing C . Hence C is definable in the pure group G . Since it is connected in the full language of the group of finite Morley rank G , it is connected in the pure group G too. Thus C is a Carter subgroup of the pure group G .

Moreover, it follows from the conjugacy of the Carter subgroups in the pure group G (Fact 2.25 (4) and Lemma 10.1) and from the paragraph above that each Carter subgroup of the pure group G is a Carter subgroup of G . \square

Proposition 10.3. – *Let G be a minimal connected simple group of finite Morley rank with a nontrivial Weyl group. Then any element x of G is unipotent if and only if it is unipotent in the pure group G .*

In particular, the unipotent elements of G are preserved by all the automorphisms of the pure group G .

PROOF – First we note that Lemma 10.1 says that the pure group G is also a minimal connected simple group of finite Morley rank with a nontrivial Weyl group.

For each $x \in G$, We denote by $d(x)$ the definable hull of $\{x\}$ relative to the pure group G , and by $d_0(x)$ its definable hull relative to the full language of G . In particular, $x \in d_0(x) \leq d(x)$, and $d(x)$ is definable in the full language of G .

We assume that x is a unipotent element of the pure group G . Then $d(x)$ contains no nontrivial semisimple element of the pure group G , and Proposition 10.2 implies that $d(x)$ contains no nontrivial semisimple element of G . Since we have $d_0(x) \leq d(x)$, the element x is unipotent in G .

Now we assume that x is unipotent relatively to G . Since the Weyl group of the pure group G is nontrivial by Lemma 10.1, Theorem 8.5 (1) says that, in the

pure group G , there exists a unique semisimple element x_s and a unique unipotent element x_u satisfying $x = x_s x_u = x_u x_s$. But the previous paragraph shows that x_u is unipotent in G too, and Proposition 10.2 provides the semisimplicity of x_s in G . Hence Theorem 8.5 (1) applied to G gives $x_s = 1$, and $x = x_u$ is unipotent in the pure group G . \square

We will need the following characterization of generics in stable groups.

Fact 10.4. – [PoizGrSt, Lemme 2.5] *Let X be a definable subset of a group G of finite Morley rank. Then X is generic in G if and only if G is covered by finitely many translations of X .*

Corollary 10.5. – *Let G be a group of finite Morley rank. Let X be a subset of a definable subgroup H of G . If X and H are definable in the pure group G , then the following two conditions are equivalent:*

- (1) *X is a generic subset of H relatively to G ;*
- (2) *X is a generic subset of H relatively to the pure group G .*

Theorem 10.6. – *Let G be a minimal connected simple group of finite Morley rank. Then the pure group G is a minimal connected simple group of finite Morley rank of the same type as G .*

PROOF – First we recall that the pure group G is a minimal connected simple group of finite Morley rank with the same Weyl group as that of G by Lemma 10.1.

We show that the generous Borel subgroups of G are the ones of the pure group G . It follows from Fact 2.37 (2) and Lemma 3.10 that the generous Borel subgroups of G (resp. of the pure group G) are precisely the maximal proper subgroups among the ones generated by some generous Carter subgroups of G (resp. of the pure group G). Moreover, it follows from Proposition 10.2 and Corollary 10.5 that the generous Carter subgroups of G are the ones of the pure group G . So the generous Borel subgroups of G are the ones of the pure group G .

If G has a Borel subgroup B generically disjoint from its conjugates, then B is a generous Borel subgroup of G by Fact 2.38. So B is a generous Borel subgroup of the pure group G by the paragraph above. Since B is generically disjoint from its conjugates relatively to G , it is generically disjoint from its conjugates relatively to the pure group G too by Corollary 10.5.

Now, if the pure group G has a Borel subgroup B generically disjoint from its conjugates, then B is a generous Borel subgroup of the pure group G by Fact 2.38. So B is a generous Borel subgroup of G by Corollary 10.5. Since B is generically disjoint from its conjugates relatively to the pure group G , it is generically disjoint from its conjugates relatively to G too by Corollary 10.5 again. This finishes the proof. \square

11. AN APPLICATION

This section somewhat deviates from the general spirit of this article. Indeed, rather than the precise structural description of a general class of simple groups of finite Morley rank, it is about a particular configuration that arises in the classification of simple groups of finite Morley rank of odd type. Nevertheless, the general line of thought developed in this article turns out to be intrinsically useful in the analysis of a very concrete classification problem and provides a conceptual streamlining in the proof of a well-known theorem in the theory of simple groups

of finite Morley rank, through the use of a particular case of one of the important ingredients of this paper, namely Theorem 3.13.

We will provide a new proof of one of the main ingredients of the analysis of strongly embedded subgroups in [BCJ07]:

Theorem 11.1. – *If G has odd type and Prüfer 2-rank at least two, then G has no strongly embedded subgroup.*

Most probably, our new proof is shorter... by a few pages. More important than the economy we may be making is the conceptuality that we are hoping to bring to one of the many complicated passages in the classification of simple groups of finite Morley rank using parts of the systematic development pursued in this article. In particular, it can be expected that the main lines of the argument presented below will be generalized to other concrete problems in the analysis of simple groups of odd type.

The notion of strong embedding was imported from finite group theory, and turned out to be almost as effective a tool as in its homeland. In order to appreciate its importance, it suffices to consult Section 10.5 of [BN94], [ABC08] or [BCJ07]. We will be content with saying that a simple group of finite Morley rank with a strongly embedded subgroup is conjectured to be isomorphic to $\mathrm{PSL}_2(K)$ where K is algebraically closed of characteristic 2, and the strongly embedded subgroups are the Borel subgroups. The following is one of the many equivalent definitions:

Definition 11.2. – *Let G be a group of finite Morley rank with a proper definable subgroup M . Then M is said to be strongly embedded in G if $I(M) \neq \emptyset$ and for any $g \in G \setminus M$, $I(M \cap M^g) \neq \emptyset$, where $I(X)$ denotes the set of involutions in X .*

Note that it follows from the definition that a strongly embedded subgroup is self-normalizing. The following is a well-known characterization:

Fact 11.3. – [BN94, §10.5] *Let G be a group of finite Morley rank with a proper definable subgroup M . Then the following are equivalent:*

- (1) M is a strongly embedded subgroup;
- (2) $I(M) \neq \emptyset$, $C_G(i) \leq M$ for any $i \in I(M)$ and $N_G(S) \leq M$ for any Sylow 2-subgroup of M ;
- (3) $I(M) \neq \emptyset$ and $N_G(S) \leq M$ for any non-trivial 2-subgroup of M .

The presence of a strongly embedded subgroup imposes strong limitations on the structure of a group of finite Morley rank the most decisive of which are the ones on involutions:

Fact 11.4. – *Let G be a group of finite Morley rank with a strongly embedded subgroup M . Then the following hold:*

- (1) *A Sylow 2-subgroup of M is a Sylow 2-subgroup of G .*
- (2) *The set $I(G)$ is a single conjugacy class in G ; the set $I(M)$ is a single conjugacy class in M .*

The following maximality principle will be useful in our proof.

Fact 11.5. – [Alt96, Proposition 3.4] *Let G be a group of finite Morley rank with a strongly embedded subgroup M . If N is a proper definable subgroup of G that contains M , then N is strongly embedded as well.*

The one ingredient that we will use from [BCJ07] is the following theorem that constitutes Case I in that article.

Fact 11.6. – [BCJ07, §4] *Let G be a minimal connected simple group of finite Morley rank and odd type. Suppose that M is a definable strongly embedded subgroup of G . Then no involution of M° lies inside $Z(M^\circ)$.*

Apart from this important ingredient, our proof will be only using [BD09]. This use necessitates a certain care as we will try to avoid any reference to [Del08] since such a reference will potentially involve an implicit use of [BCJ07] and thus, a vicious circle. This special care is the reason why we verify towards the end of the proof that the Weyl group is of odd order. As was explained in Section 3, in this particular case Theorem 3.13 has a direct proof using the relevant parts of [BD09] that do not use [Del08].

The rest of the ingredients are of a more general nature around genericity arguments such as Fact 2.17. and the following “covering” statement:

Fact 11.7. – [AB09, Corollary 4.4] *Let G be a minimal connected simple group of finite Morley rank. If x is an element of finite order then x lies inside any Borel subgroup containing $C_G(x)^\circ$.*

PROOF OF THEOREM 11.1 – By way of contradiction, let us suppose that G contains a strongly embedded subgroup M . By Fact 11.5, we may assume M is a maximal strongly embedded subgroup G .

By Fact 2.17, every involution in a group of odd type is contained in the connected component of its centralizer. It then follows from Facts 11.3 and 11.4 (2) that $I(M) \subseteq M^\circ$.

Next, we prove that M is not connected. Since G is minimal connected simple, M° is solvable. It follows from the previous paragraph and Fact 11.4 (2) that either $I(M) \subseteq F(M^\circ)$ or $I(M) \subseteq M^\circ \setminus F(M^\circ)$. The quotient $M^\circ/F(M^\circ)$ is abelian by Fact 2.12. Moreover, since $I(F(M^\circ)) = \emptyset$, $I(M^\circ/F(M^\circ)) = I(M^\circ)F(M^\circ)/F(M^\circ)$, and it follows using Fact 11.4 (2) that $I(M^\circ/F(M^\circ))$ is a conjugacy class under the action of M . If, on the other hand, $I(F(M^\circ)) \neq \emptyset$ has involutions then these belong to the connected component of the unique of the unique Sylow 2-subgroup of $F(M^\circ)$, and hence they are central in M° . Thus in both cases, involutions in $M^\circ/F(M^\circ)$ or $F(M^\circ)$ cannot be conjugated by elements of M° . But they are conjugate under the action of M . Since in both cases there are at least involutions to be conjugated by the assumption on the Prüfer 2-rank of G , we conclude that $M > M^\circ$.

Let T be a maximal decent torus of M . It is also a maximal decent torus of G since T contains the connected component of a Sylow 2-subgroup of G whose normalizer is contained in M by Fact 11.3 (3). In particular, $N_G(T) \leq M$ as well. By a Frattini argument using Fact 2.23 (1), we conclude that $M = N_G(T)M^\circ$. The above paragraph and Fact 2.24 (3) imply that $W(G) \neq 1$.

By Fact 11.6, $I(Z(M^\circ)) = \emptyset$. It follows that $I(F(M^\circ)) = \emptyset$. In particular, M° is not nilpotent. We will prove that M° is a Borel subgroup of G . Suppose that $M^\circ \leq B$ where B is a Borel subgroup of G . Then, for $w \in M \setminus M^\circ$, $M^\circ \leq B \cap B^w$. Since M° is not nilpotent, we can apply Fact 2.36 (1). Let $H = (B \cap B^w)^\circ$. Then $M^\circ \leq H$ and $M^{\circ'} \leq H'$. Thus, $C_G(H') \leq C_G(M^{\circ'})$. By the maximal choice of M with respect to strongly embedded subgroups of G , $C_G(M^{\circ'}) \leq N_G(M^{\circ'}) = M$. Since M° is not nilpotent, Fact 2.36 (1) implies that $B = B^w$. Since w was

arbitrarily chosen from $M \setminus M^\circ$, we conclude that $M \leq N_G(B)$. It follows that $M^\circ = B$.

Next, we show that M/M° is of odd order. Suppose towards a contradiction that the finite quotient M/M° is of even order. Then, there is a non-trivial 2-element $x \in M \setminus M^\circ$. Since M is strongly embedded, Fact 11.3 implies $C_G(x) \leq N_G(\langle x \rangle) \leq M$. In particular, $C_G(x)^\circ \leq M^\circ$ and the previous paragraph shows that M° is a Borel subgroup of G . By Fact 11.7 $x \in M^\circ$, a contradiction to the choice of x .

Now, we can finish the argument. The subgroup $C_G(T)$, which is connected by Fact 2.24 (3) contains a Carter subgroup of M° . By Fact 2.25 (6), $M^\circ = C_G(T)F(M^\circ)$. Since $M = N_G(T)F(M^\circ)$, it follows using the above paragraph and the conclusion $I(F(M^\circ)) = \emptyset$ (Fact 11.6) that $W(G)$ has no involutions. Also, M° is a generous Borel since it contains $N_G(T)$. So $W(G) \neq 1$, which contradicts Theorem 3.13. As $W(G)$ has odd order, the note preceding Theorem 3.13 shows that we have avoided applications of [Del08] and hence [BCJ07] as well. \square

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